TM 11-684

DEPARTMENT OF THE ARMY TECHNICAL MANUAL

PRINCIPLES AND APPLICATIONS OF MATHEMATICS FOR COMMUNICATIONS-ELECTRONICS

This copy is a reprint which includes current pages from Change 1.

HEADQUARTERS, DEPARTMENT OF THE ARMY
OCTOBER 1961

CHANGE No. 1

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DEPARTMENT OF THE ARMY
WASHINGTON, D. C., *24 July 1967*

PRINCIPLES AND APPLICATIONS OF MATHEMATICS FOR COMMUNICATIONS-ELECTRONICS

TM 11-684, 6 October 1961, is changed as follows:

Page	Paragraph	Line	Manual now reads—	Change to read—
4	5, *Example 1*	4	6.25	.625
6	12*g* (10)		$1\frac{7}{8}$	$\frac{7}{8}$
12	24*b*	9	from 1 to 100	from 1 to 1,000
17	31	2	arithmetic (ch. 5).	arithmetic.
18	Caption of figure 6		*(−1 and −5).*	*(−2 and −3).*
21	42*b*, *Example 2*	4	$-(x^3 - 3x^2 - x + 1)$	$-(x^3 + 3x^2 + x - 1)$
22	45, *Example 1*	2	$(5^3)^4 = 5^{3.4} = 5^{12}$	$(5^3)^4 = 5^{3 \cdot 4} = 5^{12}$
22	45, *Example 2*	3	or $2^{1.3}a^{1.3}b^{1.3} = 8a^3b^3$	or $2^{1 \cdot 3}a^{1 \cdot 3}b^{1 \cdot 3} = 8a^3b^3$
22	45, *Example 3*	2	$(ax^2)^3 = a^{1.3}x^{2.3} = a^3x^6$	$(ax^2)^3 = a^{1 \cdot 3}x^{2 \cdot 3} = a^3x^6$
22	45, *Example 4*	2	$[(x^5)^4]^3 = [x^{5.4}]^3 = [x^{20}]^3 = x^{20.3} = x^{60}$	$[(x^5)^4]^3 = [x^{5 \cdot 4}]^3 = [x^{20}]^3 = x^{20 \cdot 3} = x^{60}$

Page	Paragraph	Line	Manual now reads—	Change to read—
22	45, *Example 5*	2	$\left(\frac{2}{x^2}\right)^5 = \frac{2^{1.5}}{x^{2.5}} = \frac{2^5}{x^{10}} = \frac{32}{x^{10}}$	$\left(\frac{2}{x^2}\right)^5 = \frac{2^{1\cdot 5}}{x^{2\cdot 5}} = \frac{2^5}{x^{10}} = \frac{32}{x^{10}}$
24	49b, *Example 3*	1	Divide $6a$	Divide $6a^2$
25	50f (5)		$(3x^2 - 2xy - 5y^2)(3x^2 + 2xy - 5y^2)$	$(3x^2 - 2xy + 5y^2)(3x^2 + 2xy - 5y^2)$
26	56a, *Example 3*	2	$2ax^2 - 4bx^2 + 6cx^2 = 2x^2\ (a - 2^b + 3^c)$	$2ax^2 - 4bx^2 + 6cx^2 = 2x^2\ (a - 2b + 3c)$
30	59g	6	$xb + xb + ab$	$xb + ab$
32	61c (8)		$\sqrt[3]{-x9}$	$\sqrt[3]{-x^9}$
32	61d (9)		$\pi r \frac{2}{1} + \pi r \frac{2}{2}$	$\pi r_1^2 + \pi r_2^2$
33	63c	3	$-\frac{(x - y)\ (x - 2y)}{x + y} = \frac{(x + y)\ (x - 2y)}{x + y} = \frac{(y - x)\ (x - 2y)}{x + y}$	$-\frac{(x - y)\ (x - 2y)}{x + y} = \frac{(-x + y)\ (x - 2y)}{x + y} = \frac{(y - x)\ (x - 2y)}{x + y}$
36	68a, *Example 1*	4	$\frac{6a^2b}{7x} \cdot \frac{21x^2y}{21a^2b} = \frac{3xy}{4}$	$\frac{6a^2b}{7x} \cdot \frac{21x^2y}{24a^2b} = \frac{3xy}{4}$

Page	Paragraph	Line	Manual now reads—	Change to read—
37	69<u>b</u> (7)	2	denominator is $2I^2R$.	denominator is $6\pi f^2c$.
38	72<u>a</u>	8	$\sqrt{2.\ 5^4} = 5^2\sqrt{2} = 25\sqrt{2}$	$\sqrt{2 \cdot 5^4} = 5^2\sqrt{2} = 25\sqrt{2}$
38	72<u>a</u>, *Example 1*	2 & 3	$\sqrt{50} = \sqrt{25.\ 2}$ $= \sqrt{25}.\ \sqrt{2}$	$\sqrt{50} = \sqrt{25 \cdot 2}$ $= \sqrt{25} \cdot \sqrt{2}$
40	73, *Example 3*	3	$= (4r)^{\frac{2}{6}} - 4(4r)^{\frac{1}{3}} + (4r)^{\frac{5}{6}}$	$= (4r)^{\frac{2}{6}} - r(4r)^{\frac{1}{3}} + (4r)^{\frac{3}{6}}$
40	73, *Example 4*		$2\sqrt{6} + 9\sqrt{\frac{2}{3}} - \frac{4}{\sqrt{36}}$ $= 2\sqrt{6} + 9\sqrt{\frac{2}{3}\frac{3}{3}} - \frac{4}{\sqrt{6.\ 6}}$ $= 2\sqrt{6} + \frac{9}{3}\sqrt{6} - \frac{4}{\sqrt{6^2}}$ $= 2\sqrt{6} + 3\sqrt{6} - \sqrt{6}$ $- 4\sqrt{6}$	$2\sqrt{6} + 9\sqrt{\frac{2}{3}} - \frac{4}{\sqrt{36}}$ $= 2\sqrt{6} + 9\sqrt{\frac{2}{3} \bullet \frac{3}{3}} - \frac{4}{\sqrt{6 \cdot 6}}$ $= 2\sqrt{6} + \frac{9}{3}\sqrt{6} - \frac{4}{\sqrt{6^2}}$ $= 2\sqrt{6} + 3\sqrt{6} - \sqrt{6}$ $= 4\sqrt{6}$
41	75<u>a</u>, *Example 1*	2	$\frac{\sqrt{15}}{\sqrt{5}} = \sqrt{\frac{5}{15}} = \sqrt{3}$	$\frac{\sqrt{15}}{\sqrt{5}} = \sqrt{\frac{15}{5}} = \sqrt{3}$

Page	Paragraph	Line	Manual now reads—	Change to read—
42	75b, Example 2	2	$\frac{\sqrt{1+z}-\sqrt{1=z}}{\sqrt{1+z}+\sqrt{1-z}}$ $\frac{\sqrt{1+z}-\sqrt{1-z}}{\sqrt{1+z}+\sqrt{1-z}}$ $\frac{\sqrt{1+z}-\sqrt{1-z}}{\sqrt{1+z}-\sqrt{1-z}}$	$\frac{\sqrt{1+z}-\sqrt{1-z}}{\sqrt{1+z}+\sqrt{1-z}} =$ $\frac{\sqrt{1+z}-\sqrt{1-z}}{\sqrt{1+z}+\sqrt{1-z}}$ $\frac{\sqrt{1+z}-\sqrt{1-z}}{\sqrt{1+z}-\sqrt{1-z}}$
42	76b (9)		$(^{2}r_1 + {}^{3}r_2)^{\frac{1}{2}}.$	$(2r_1 + 3r_2)^{\frac{1}{2}}$
43	76g (10)		$5 \; i+3$ $\sqrt{(i+3)^2}$	$5\frac{i+3}{\sqrt{(i+3)^2}}$
43	76h (4)		$\frac{3xy}{\sqrt{xu}}$	$\frac{3xy}{\sqrt{xy}}$
43	76h (5)		$\frac{2}{\sqrt{6-2}}$	$\frac{2}{\sqrt{6-2}}$
43	76h (9)		$\frac{\sqrt{e^2+f^2+f}}{\sqrt{e^2+f^2-f}}$	$\frac{\sqrt{e^2+f^2}+f}{\sqrt{e^2+f^2}-f}$

Page	Paragraph	Line	Manual now reads—	Change to read—
44	77*b*	7	$j^5 = j \cdot j \cdot j \cdot j \cdot j = j^4 \cdot j = j^1 = \sqrt{-1}$	$j^5 = j \cdot j \cdot j \cdot j \cdot j = j^4 \cdot j = j \cdot 1 = \sqrt{-1}$
44	77*c*, *Example 2*	2-5	$6\sqrt{-2} + 5\sqrt{-8} + 8\sqrt{-18}$ $= j^6\sqrt{2} + j^5\sqrt{8} + j^8\sqrt{18}$ $= j^6\sqrt{2} + j(5 \cdot 2)\sqrt{2} + j(8 \cdot 3)\sqrt{2}$ $= (j^6 + j^{10} + j^{24})\sqrt{2}$ $= j^{40}\sqrt{2}$	$6\sqrt{-2} + 5\sqrt{-8} + 8\sqrt{-18}$ $= j6\sqrt{2} + j5\sqrt{8} + j8\sqrt{18}$ $= j6\sqrt{2} + j(5 \cdot 2)\sqrt{2} + j(8 \cdot 3)\sqrt{2}$ $= (j6 + j10 + j24\sqrt{2}$ $= j40\sqrt{2}$
44	77*c*, *Example 3*	2	$\sqrt{-36} - \sqrt{-64} = j^6 - j^8 = -j^2$	$\sqrt{-36} - \sqrt{-64} = j6 - j8 = -j2$
44	77*c*, *Example 4*	3 & 4	$= (j^{18} - j^8)\sqrt{2}$ $= j^{10}\sqrt{2}$	$= (j18 - j8)\sqrt{2}$ $= j10\sqrt{2}$
45	77*d*, *Example 1*	2	$\sqrt{-16} \cdot \sqrt{-4} = j^4 \cdot j^2 = j^2 8 = (-1)8 = -8$	$\sqrt{-16} \cdot \sqrt{-4} = j4 \cdot j2 = j^2 8 = (-1)8 = -8$
45	77*d*, *Example 2*	2	$\sqrt{-81} \cdot \sqrt{-25} \cdot \sqrt{-49}$ $= j^9 \cdot j^5 \cdot j^7$ $= j^3 315$ $= (-j)315$ $= -j315$	$\sqrt{-81} \cdot \sqrt{-25} \cdot \sqrt{-49}$ $= j9 \cdot j5 \cdot j7$ $= j^3 315$ $= (-j)315$ $= -j315$

Page	Paragraph	Line	Manual now reads—	Change to read—
45	77g, *Example 4*	2	$\frac{6}{j} = \frac{6}{j} \cdot \frac{j}{j} = \frac{j6}{j2} = \frac{j6}{-1} = -j6$	$\frac{6}{j} = \frac{6}{j} \cdot \frac{j}{j} = \frac{j6}{j^2} = \frac{j6}{-1} = -j6$
47	78d, *Example 1*	4	$= \frac{8 + j - (1 - 1)4}{1 - (-1)}$	$= \frac{8 + j - (-1)4}{1 - (-1)}$
62	86g (9)	2	$\frac{8}{x} + \frac{1}{y} = 1\frac{3}{5}$	$\frac{8}{x} - \frac{1}{y} = 1\frac{3}{5}$
62	102g (4)	1	$f(x) - x^2 - 6x + 12$	$f(x) = x^2 - 6x + 12$
70	118, *Example 2* *Step 3*	5	.0021	.00121
71	119, *Example 2*	1	6.7166 — 10	6.8166 — 10
71	119, *Example 2* *Step 6*	4	6.7166 — 10	6.8166 — 10
84	140g, *Example 2*	10	$D = \sqrt[2]{\frac{A}{\pi}}$	$D = 2\sqrt{\frac{A}{\pi}}$

Page	Paragraph	Line	Manual now reads—	Change to read—
84	140g, *Example 2*	12 & 13	$D = \sqrt[2]{\frac{78.54}{3.1416}}$ $D = \sqrt[2]{25}$	$D = 2\sqrt{\frac{78.54}{3.1416}}$ $D = 2\sqrt{25}$
89	147, *Example 1*	1	Given the right triangle *ABC* (fig. 23):	Given the right triangle *ABC* (fig. 34):
90	147, *Example 2*	1	Given the right triangle *ABC* (fig. 23): side *A* is $\sqrt{5}$;	Given the right triangle *ABC* (fig. 34): side *a* is $\sqrt{5}$;
91	148, *Example 1*	15	$\cos A = \frac{\sqrt{6}}{30} = \frac{1}{5}$	$\cos A = \frac{6}{30} = \frac{1}{5}$
95	152d, *Example 1*	3	$\sqrt[3]{71° 22' 21''}$	$3\overline{)71° 22' 21''}$
95	152d, *Example 2*	3	$\sqrt[6]{166° 17' 36''}$	$6\overline{)166° 17' 36''}$
106	164g (4)	1	Side *B*	Side *b*
107	165	1	An oblique triangle is one in which one of	An oblique triangle is one in which none of
122	175g	4	$40° = \frac{1}{4}\ \pi$ radians	$45° = \frac{1}{4}\ \pi$ radians

Page	Paragraph	Line	Manual now reads—	Change to read—
135	192h	4	cuit *a*	cuit *A*
160	204f			Delete the first sentence and insert the following: A 300-volt, 60-cycle ac generator is connected in series with a 6-ohm resistance, and 8-ohm inductive reactance, and a 16-ohm capacitive reactance.
170	224	25	(par. 257).	(par. 223).
232	21f	2	2,567 pounds	2.567 pounds
232	42g (3)		$\frac{3}{35}$	$-\frac{3}{35}$
232	42g (5)		$-\frac{2}{3}$ ampere	$\frac{2}{3}$ ampere
232	50g (1)		$4a^4 - 4a^2b^2 + 4b^4$	$6a^4 - 4a^2b^2 + 4b^4$
232	50g (2)		$\frac{1}{r^3g^4}$	$\frac{1}{r^2x^4}$
233	50f (6)		$\frac{a-e}{ex}$	$\frac{a-e}{ex}$

Page	Paragraph	Line	Manual now reads—	Change to read—
223	61c (2)		-8	8
223	61c (5)		$-10ab^5$	$\pm 10ab^5$
223	61d (7)		14^2ry	$14ry^2$
223	69c (4)		$x = -1\frac{1}{12}$	$x = -1\frac{13}{16}$
223	69c (7)		$\frac{y}{C^2 - d^2}\quad\frac{x(c + d)}{C^2 - d^2}$	$\frac{y}{c^2 - d^2},\ \frac{x(c + d)}{c^2 - d^2}$
223	69d (4)		$\frac{6(x^2 - 2)}{x^4 - 5x^2 + 14}$	$\frac{6(x^2 - 2)}{x^4 - 5x^2 + 4}$
232	69d (5)		$\frac{9c + 2cd \div 12d}{12c^2d^2}$	$\frac{9c^2 + 2cd - 12d^2}{12c^2d^2}$
233	69d (10)		$\frac{3t + 4y}{12tv^2}$	$\frac{3t + 4v}{12tv^2}$
233	76b (4)		$\frac{2}{\sqrt{27}}$	$2\sqrt{27}$
233	76b (5)		$\frac{5}{\sqrt{x}}$	$5\sqrt{x}$

Page	Paragraph	Line	Manual now reads—	Change to read—
233	76b (6)		$\overset{4}{\sqrt{a^2b^6}}$	$\sqrt[4]{a^2c^6}$
233	76b (8)		$\overset{3}{2b}\sqrt{a^3}$	$2b\sqrt[3]{a^3}$
233	76c (3)		$\frac{\overset{2}{\sqrt{5a}}}{3}$	$\frac{2\sqrt{5a}}{3}$
234	76h (9)		$\frac{e^2 + f^2 + 2f\sqrt{e^2 + f^2}}{e^2}$	$\frac{e^2 + 2f^2 + 2f\sqrt{e^2 + f^2}}{e^2}$
234	76f (7)		$\overset{2}{\sqrt{5}} + 3\sqrt{x}$	$\sqrt{5} + 3\sqrt{x}$
234	79b (4)		$4 + j10$	$44 + j10$
234	79d (7)		$f^2 + jfg - g^2$	$f^2 + 2jfg - g^2$
234	79g (5)		$\frac{x^2 + j^2xy - y^2}{x^2y^2}$	$\frac{x^2 + 2jxy - y^2}{x^2 + y^2}$
234	79g (10)		$\frac{I^2 + j2IE - E^2}{I^2 \quad E^2}$	$\frac{I^2 + j2IE - E^2}{I^2 + E^2}$

Page	Paragraph	Line	Manual now reads—	Change to read—
234	86*c* (5)		$\frac{28}{9}$	24 9
235	94*c* (4)		$\frac{3}{2}, \frac{4}{3}$	$-\frac{3}{2}, \frac{4}{3}$
235	127*f* (4)		83. 28	33. 37
235	153*b* (1)		$\tan A = \frac{4}{\sqrt{33}}$	$\tan A = \frac{4}{33}\sqrt{33}$
236	164*c* (11)		32. 9	30.9
236	164*d* (12)		side opposite 60° /5. 196 inches, side opposite 30° /3 inches.	side opposite /60° 5.196 inches, side opposite /30° 3 inches.

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For explanation of abbreviations used, see AR 320-50.

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HEADQUARTERS,
DEPARTMENT OF THE ARMY
WASHINGTON 25, D.C., *6 October 1961*

PRINCIPLES AND APPLICATIONS OF MATHEMATICS FOR COMMUNICATIONS-ELECTRONICS

PART I

MATHEMATICAL PRINCIPLES ESSENTIAL TO COMMUNICATIONS-ELECTRONICS

CHAPTER 1

INTRODUCTION

1. Purpose and Scope

a. Purpose. This manual provides the basic mathematics required by communications-electronics personnel.

b. Scope. This manual covers those principles and applications of arithmetic, algebra, logarithms, geometry, and trigonometry that are required for a practical understanding of electricity and electronics. The manual is divided into two parts:

(1) Part I is a review of the mathematical principles essential to communications-electronics.

(2) Part II covers the application of the mathematical principles to common communications-electronics problems.

2. Mathematics and Electronics

Skill in the use of mathematics, particularly arithmetic, algebra, and trigonometry, is essential in the fields of electricity and electronics. Most of our basic ideas of electrical phenomena are based on mathematical reasoning and are stated in mathematical terms. Therefore, a thorough knowledge of mathematics and of the specific applications of mathematics to the field of electricity will serve as a foundation for the technical knowledge needed by communications-electronics personnel.

CHAPTER 2
PERCENTAGE

3. General

a. Definition. Percentage is the process of computation in which the basis of comparison is a *hundred.* The term *percent*—from *per,* by, and *centum,* hundred—means *by* or *on the hundred.* Thus, 2 percent of a quantity means two parts of every hundred parts of the quantity.

b. Symbol. The symbol of percentage is %. Percent may also be indicated by a fraction or a decimal. Thus, $5\% = \frac{5}{100} = .05$. Figure 1 shows the relationship between fractions, decimals, and percentage.

c. Base, Rate, and Percentage.

(1) The *base* is the number on which the percentage is computed.

(2) The *rate* is the amount (in hundredths) of the base to be estimated.

(3) The *percentage* is a part or proportion of a whole expressed as so many per hundred. Percentage is the portion of the base determined by the rate.

4. Conversion of Decimal to Percent

To change a decimal to percent, move the decimal point two places to the right and add the percent symbol.

Example: Chance .375 to percent.
Move decimal point two places to right: 37.5
Add percent symbol: 37.5%

5. Conversion of Fraction to Percent

To convert a fraction to percent, divide the numerator by the denominator and convert to a decimal. Then, convert the decimal to percent (par. 4).

Example: Change fraction $\frac{5}{8}$ to percent.
Divide numerator by denominator: $5 \div 8 = .625$
Convert decimal to percent: 6.25 = 62.5%
Thus, $\frac{5}{8} = 62.5\%$.

6. Conversion of Percent to Decimal

To change a percent to a decimal, omit the percent symbol and move the decimal point two places to the left.

Example 1: Change 15% to a decimal.
Omit percent symbol: 15% becomes 15
Move decimal point two places to the left: 15 becomes .15
Thus, 15% = .15.

Example 2: Change 110% to a decimal.
Omit percent symbol: 110% becomes 110
Move the decimal point two places to the left: 110 becomes 1.10.
Thus, 110% = 1.10.

7. Conversion of Percent to Fraction

To change a percent to a fraction, first change the percent to a decimal (par. 6) and then to a fraction. Reduce the fraction to its lowest terms.

Example 1: Change 25% to a fraction.
Change to a decimal: 25% = .25
Change to a fraction: $.25 = \frac{25}{100}$
Reduce fraction to lowest terms: $\frac{25}{100} = \frac{1}{4}$
Thus, $25\% = \frac{1}{4}$.

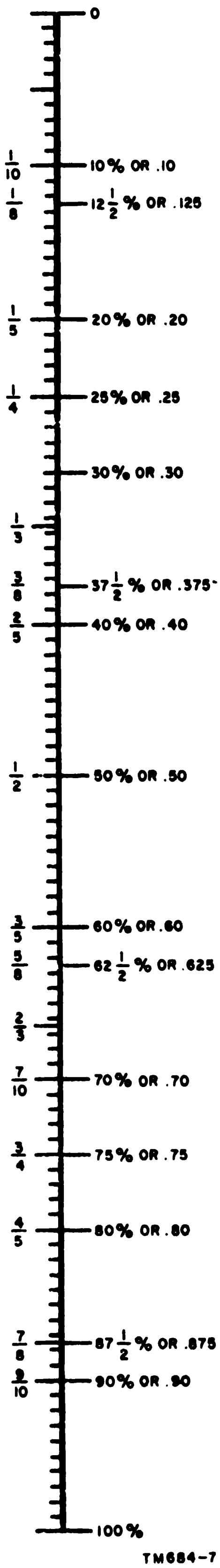

Figure 1. Relationship between fractions, decimals, and percentage.

Example 2: Change 37.5% to a fraction.
Change to a decimal: 37.5% = .375
Change to a fraction:

$$.375 = \frac{375}{1000}$$

Reduce fraction to lowest terms:

$$\frac{375}{1000} = \frac{3}{8}$$

Thus, $37.5\% = \frac{3}{8}$.

8. Finding Percentage

a. General. To find the percent of a number, write the percent as a decimal and multiply the number by this decimal. In this case, the ***base*** and ***rate*** are given. The problem is to find the percentage.

Example 1: Find 5% of 140 (140 is the base, 5% is the rate, and the product is the percentage).
$5\% \text{ of } 140 = .05 \times 140 = 7$

Example 2: Find 5.2% of 140.
$5.2\% \text{ of } 140 = .052 \times 140 = 7.28$

Example 3: Find 150% of 36.
$150\% \text{ of } 36 = 1.50 \times 36 = 54$

Example 4: Find $\frac{1}{2}\%$ of 840.
$\frac{1}{2}\% = .5\%$
$.5\% \text{ of } 840 = .005 \times 840 = 4.20$
Thus, $\frac{1}{2}\% \text{ of } 840 = 4.20$.

b. Application of Percentage. In communications-electronics, typical applications of percentage computation are used in determining tolerance values of resistors (par. 206) or in determining the efficiencies of motors and generators (par. 209).

9. Finding Rate

To find the percent one number is of another, write the problem as a fraction, change the fraction to a decimal, and write the decimal as a percent. In this case, the *percentage* and *base* are given. The problem is to find the *rate*.

Example 1: 3 is what percent of 8? (3 is the percentage, 8 is the base, and the quotient is the rate.)

$$\frac{3}{8} = .375$$

$.375 = 37.5\% = 37\frac{1}{2}\%$

Therefore, 3 is $37\frac{1}{2}\%$ of 8.

Example 2: What percent of 542 is 234?

$\frac{234}{542} = .4317 +$ (round off)

$.432 = 43.2\%$

Therefore, 234 is 43.2% of 542.

Example 3: 125 is what percent of 50?

$\frac{125}{50} = 2.50$

$2.50 = 250\%$

Therefore, 125 is 250% of 50.

10. Finding Base Numbers

To find a number when a percent of the number is known, first find 1% of the number, and then find 100% of the number. In this case, the *percentage* of the number and the *rate* are given. The problem is to find the *base*.

Example 1: 42 is 12% of what number?

12% (base number) = 42

1% (base number) =

$\frac{42}{12} = 3.50$

100% (base number) =

$100 \times 3.50 = 350$

Therefore, the base number is 350.

Example 2: 45 is 150% of what number?

150% (base number) = 45

1% (base number) $= \frac{45}{150} = .3$

100% (base number) =

$100 \times .3 = 30$

Therefore, the base number is 30.

11. Expressing Accuracy of Measurements in Percent

a. Relative error is the accuracy of a measurement expressed in percent of the total measurement. In determining the relative error, it is first necessary to establish the *limit of error.*

b. The *limit of error* is the difference between the *true value* and the *measured value.* Assume that the reading on a scale, to the nearest tenth of an inch, is 2.2 inches. If the true value is 2.15 inches, the limit of error is the difference between 2.15 and 2.20, or .05 inch.

c. Relative error is computed by solving the ratio $\frac{\text{LIMIT OF ERROR}}{\text{MEASURED VALUE}}$, and expressing the result as a percent. In the scale reading above, the relative error $= \frac{.05}{2.2} = 2.27\%$, or 2.3%.

12. Review Problems—Percentage

a. Show each of the following in three forms—as a fraction or mixed number, as a decimal, and as a percent:

(1) $\frac{3}{5}$

(2) 50%

(3) .375

(4) $\frac{1}{4}$

(5) $62\frac{1}{2}\%$

(6) .6

(7) $\frac{3}{10}$

(8) 70%

(9) 2.25

(10) $1\frac{7}{8}$

(11) .08

(12) $\frac{3}{50}$

(13) .18

(14) $\frac{1}{4}\%$

(15) .025

(16) .05

(17) $8\frac{1}{3}\%$

(18) $37\frac{1}{2}\%$

(19) 105%

(20) 4%

b. Evaluate the following:

(1) 250% of 60

(2) 125% of 40

(3) 200% of 2

(4) 225% of 400

c. What percent of a number is—

(1) 1.5 times the number?

(2) $2\frac{3}{4}$ times the number?

(3) $\frac{3}{2}$ times the number?

(4) $5\frac{1}{2}$ times the number?

d. Find the following:

(1) $\frac{2}{5}\%$ of 410

(2) $\frac{3}{5}\%$ of 416,000

(3) $\frac{2}{5}\%$ of 85

(4) 5.2% of 85

e. Solve the following problems:

(1) Find the relative error for a limit of error of .05 inch in measuring 24.2 inches.

(2) Find the relative error for a limit of error of 2 inches in measuring 200 yards.

f. Find the number when—

(1) 12% of the number is 52

(2) 15% of the number is 375

(3) 32% of the number is 166.4

(4) 8% of the number is 16

(5) 84% of the number is 168

CHAPTER 3
RATIO AND PROPORTION

Section I. RATIO

13. Understanding Ratio

It is often desirable, for the purpose of comparison, to express one quantity in terms of another quantity of the same kind. One way to express this relationship is by means of a *ratio*. For example, if one resistor has a resistance of 800 ohms and another has a resistance of 100 ohms, the first resistor has 8 times as much resistance as the second. In other words, the ratio between the resistors is 8 to 1.

14. Expressing Ratio

Ratio can be expressed in four different ways For example, the ratio of 12 to 3 can be expressed as follows: 12 to 3, 12:3, 12 ÷ 3, or $\frac{12}{3}$. The numbers 12 and 3, which are the terms of the ratio, are called the *antecedent* and the *consequent*, respectively. The antecedent is the dividend or the numerator; the consequent is the divisor or denominator.

15. Obtaining Value of Ratio

Both terms of any ratio may be multiplied and divided by the same number without changing the value of the expression. In the ratio $\frac{12}{3}$, for example, the 12 is divided by 3, giving the value of 4. This means that the ratio 12:3 is equal to the ratio 4:1.

Example 1: What is the ratio of 6:2?

$\frac{6}{2} = 3$, or 3:1

Example 2: What is the ratio of 7:3?

$\frac{7}{3} = 2\frac{1}{3}$ or $2\frac{1}{3}$:1

Example 3: Find the ratio of the areas (par. 26) of two squares the sides of which are 6 and 8 inches, respectively. The areas of similar figures are in the same ratios as the squares of their like dimensions.

$8^2 : 6^2 = 64 : 36$

$\frac{64}{36} = 1\frac{28}{36} = 1\frac{7}{9}$ or $1\frac{7}{9}$:1

Thus, the second square (8 inches on a side) is $1\frac{7}{9}$ times as large as the first square (6 inches on a side).

Section II. PROPORTIONS

16. Understanding Proportion

A proportion is a statement of equality between two ratios. If the value of one ratio is equal to the value of another ratio, they are said to be in proportion. For example, the ratio 3:6 is equal to the ratio 4:8. Therefore, this can be written 3:6 :: 4:8 or 3:6 = 4:8. In any proportion, the first and last terms are called the *extremes;* the second and third terms are called the *means* (fig. 2).

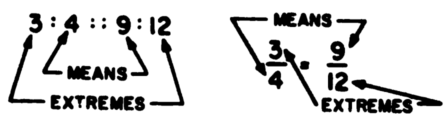

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Figure 2. Terms of proportion.

17. Rules of Proportion

There are three rules of proportion that are used in determining an unknown quantity.

They also can be used to prove that the proportion is true.

a. In any proportion, *the product of the means equals the product of the extremes.*

Example 1: 3:4 :: 9:12.

$3 \times 12 = 36$ (product of extremes)

$4 \times 9 = 36$ (product of means)

Example 2: $\frac{3}{4} = \frac{9}{12}$.

Note. When the proportion is expressed in fractional form, the numerator of one fraction is multiplied by the denominator of the other fraction. This process is called *cross-multiplication.*

$3 \times 12 = 36$ (product of extremes)

$4 \times 9 = 36$ (product of means)

b. In any proportion, *the product of the means divided by either extreme gives the other extreme.*

Example: 6:8 :: 18:24.

$8 \times 18 = 144$ (product of means)

$144 \div 6 = 24$ (one extreme)

$144 \div 24 = 6$ (other extreme)

c. In any proportion, *the product of the extremes divided by either mean gives the other mean.*

Example: 5:7 :: 15:21

$5 \times 21 = 105$ (product of extremes)

$105 \div 7 = 15$ (one mean)

$105 \div 15 = 7$ (other mean)

18. Solving for Unknown Term

As demonstrated in paragraph 49, the unknown term of a proportion can be determined if the other three terms are known.

Example 1: In the proportion $\frac{5}{10} = \frac{10}{y}$, solve for y (the unkown quantity).

Find the product of the means: $10 \times 10 = 100$

Find the product of the extremes: $5 \times y = 5y$

The products of the means and extremes are equal: $5y = 100$

Divide both sides by 5:

$$\frac{\cancel{5}y}{\cancel{5}} = \frac{\overset{20}{\cancel{100}}}{\cancel{5}}$$

$y = 20$

Therefore, $\frac{5}{10} = \frac{10}{20}$

Example 2: In the proportion 6:12 :: 24:y, solve for y.

Write the proportion in fractional form:

$$\frac{6}{12} = \frac{24}{y}$$

Cross-multiply.

$6y = 288$

Divide both sides by 6.

$$\frac{\cancel{6}y}{\cancel{6}} = \frac{\overset{48}{\cancel{288}}}{\cancel{6}}$$

$y = 48$

Therefore, 6:12 :: 24:48.

Example 3: In the proportion $\frac{z}{20} = \frac{5}{10}$, solve for z.

Cross-multiply.

$10z = 100$

Divide both sides by 10:

$$\frac{\cancel{10}z}{\cancel{10}} = \frac{\overset{10}{\cancel{100}}}{\cancel{10}}$$

$z = 10$

Therefore, $\frac{10}{20} = \frac{5}{10}$.

19. Stating Ratios for Problems in Proportion

When setting up a proportion problem, be sure to state the ratios correctly. Analyze each problem carefully to determine whether the unknown quantity will be greater or lesser than the known term of the ratio in which it occurs. Arrange the terms of the ratio as shown below, and solve for the unknown quantity as explained in paragraph 18.

$$\frac{\text{LESSER}}{\text{GREATER}} = \frac{\text{LESSER}}{\text{GREATER}}, \text{ or LESSER : GREATER :: LESSER : GREATER}$$

Example: The weight of 15 feet of iron pipe is 8 pounds.

What is the weight of 255 feet of the same pipe? Let the unknown quantity be represented by the letter y. Since ratios must express a relation between quantities of the same kind, one ratio must be between feet and feet and the other between pounds and pounds.

Study the problems; 255 feet of pipe will weigh more than 15 feet of pipe. Arrange the first ratio in the order LESSER to GREATER—15 feet: 255 feet, or $\frac{15}{255}$.

Arrange the second ratio in the same order—LESSER to GREATER—8 pounds: y pounds, or $\frac{8}{y}$.

Write the proportion and solve.

$15:255 = 8:y$, or

$$\frac{15}{255} = \frac{8}{y}$$

$$15y = 255 \times 8$$

$$15y = 2040$$

$$y = \frac{2040}{15}$$

$$y = 136 \text{ pounds}$$

20. Inverse Proportion

a. The ratio 2:3 is the inverse of the ratio 3:2. In proportion, when a second ratio is equal to the inverse of the first ratio, the elements are said to be *inversely proportional.*

b. Two numbers are inversely proportional when one increases as the other decreases. In this case, their product is always the same. In problems dealing with pulleys, the speeds of different size pulleys connected by belts are inversely proportional to their diameters. A smaller pulley rotates faster than a larger pulley.

Example 1: **A pulley 30 inches in diameter is turning at a speed of 300 revolutions per minute. If this pulley is belted to a pulley 15 inches in diameter (fig. 3), determine the speed at which the smaller pulley is turning.**

Let the speed of the smaller pulley be represented by y. Study the problem; the first ratio will be between inches and the second will be between revolutions per minute (rpm). Also note that the second pulley is smaller than the first and must make more revolutions than the first. Therefore, the answer will be a number larger than 300.

Arrange the ratios in the order LESSER to GREATER.

First ratio:

$15:30$, or $\frac{15}{30}$

Second ratio:

$300:y$, or $\frac{300}{y}$

The proportion:

$15:30 = 300:y$, or $\frac{15}{30} = \frac{300}{y}$

Solve the proportion:

$$\frac{15}{30} = \frac{300}{y}$$

$$15y = 300 \times 30$$

$$15y = 9000$$

$$y = \frac{9000}{15}$$

$$y = 600 \text{ rpm}$$

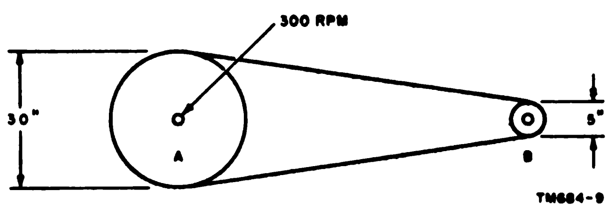


Figure 3. Pulleys and inverse ratio.

Example 2: A 24-inch pulley is fixed to a drive shaft that is turning at the rate of 400 rpm. This pulley is belted to a 6-inch pulley. Determine the speed of the smaller pulley in revolutions per minute.

Driving pulley (400 rpm, 24 inches in diameter).

Driven pulley (y rpm, 6 inches in diameter).

$$\frac{6}{24} = \frac{400}{y}$$

$$6y = 400 \times 24 = 9{,}600$$

$$y = 1{,}600 \text{ rpm}$$

21. Problems Using Proportion

a. A steel plate ½ inch thick, 12 inches wide, and 9 feet long weighs 183.6 pounds. What is the weight of a piece of steel plate of the same thickness and width if it is 16 feet 6 inches long?

b. If three men complete a certain job in 8 days, how many days would it take seven men to complete the same job, considering that they will work at the same speed?

c. If 3 resistors cost 25 cents, find the cost of 60 resistors at the same rate?

d. If the upkeep on 62 trucks for a year is $3,100, what would be the upkeep on 28 such trucks for 1 year at the same rate?

e. At a given temperature, the resistance of a wire increases with its length. If the resistance of a wire per 1,000 feet at 68°F is .248 ohm, what is the resistance of 1,500 feet; of 1,200 feet; of 1,850 feet; of 3,600 feet?

f. If 21-gage wire weighs 2.452 pounds per 1,000 feet, what is the weight of 1,150 feet; 1,540 feet; 1,680 feet; 349 yards?

g. The speeds of gears running together are inversely proportional to the number of teeth in the gears. A driving gear with 48 teeth meshes with a driven gear with 16 teeth. If the driving gear turns at the rate of 100 rpm, how many rpm are made by the driven gear?

h. A 36-tooth gear running at a speed of 280 rpm drives another gear with 64 teeth. What is the speed of the other gear?

CHAPTER 4
POWERS AND ROOTS

22. Powers

There are many times in mathematics when a number must be multiplied by itself a number of times, such as $4 \times 4 \times 4 \times 4 \times 4$. This is written as 4^5 and is described as 4 raised to the fifth power. A number multiplied by itself once is said to be raised to the second power (squared). Thus, 5×5 is written 5^2. The number 2, written to the right and above the number 5, is the *exponent;* the number 5 is the *base*. The base number is a *factor* of a number written in exponential form because the product is evenly divisible by the base.

23. Roots

The root of a number is that number which, when multiplied by itself a given number of times, will equal the given number. The square root of 25 is 5, since 5×5 or 5^2 equals 25. The third root (cube root) of 216 is 6, since $6 \times 6 \times 6$ or 6^3 equals 216. The fourth root of 81 is 3, since $3 \times 3 \times 3 \times 3$ or 3^4 equals 81. Extraction of a root is generally indicated by placing, in front of the number, a *radical sign* ($\sqrt{\quad}$). A small figure is placed in the angle at the front of the sign to indicate the root to be taken. If the small figure is omitted, it is understood that the operation required is square root.

Thus,

$$\sqrt{25} = 5$$
$$\sqrt[3]{216} = 6$$
$$\sqrt[4]{81} = 3$$

24. Finding Square Root of a Number

a. Finding Square Root by Mental Calculation. In some instances, the square root can be determined mentally from a knowledge of common multiplication. For example, $\sqrt{25}$ is 5, since 5×5 or $5^2 = 25$. Similarly, $\sqrt{144}$ is 12, since 12×12 or $12^2 = 144$.

b. Finding Square Root by Arithmetical Process. In most cases, the square root of a number must be determined by a mathematical process. If the number is a perfect square, the square root will be an integral number; if the number is not a perfect square, the square root will be a continued decimal. To save time in calculation, a table of square roots of numbers from 1 to 100 is given in appendix III.

Example 1: Evaluate $\sqrt{3398.89}$.

Step 1. Starting at the decimal point mark off the digits in pairs in both directions.

$$\sqrt{33\ 98.89}$$

Step 2. Place the decimal point for the answer directly above the decimal point that appears under the radical sign.

$$\sqrt{33\ 98.89}$$

Step 3. Determine by inspection the largest number that can be squared without exceeding the first pair of digits—33. The answer is 5, since the square of any number larger than 5 will be greater than 33. Place the 5 above the first pair of digits.

$$\begin{array}{l} \ \ 5 \\ \sqrt{33\ 98.89} \end{array}$$

Step 4. Square 5 to obtain 25, and place it under 33. Substract 25 from 33 and obtain 8. Bring down the next pair of digits—98.

$$\begin{array}{l} \ \ 5 \\ \sqrt{33\ 98.89} \\ \ \ \underline{25} \\ \ \ \ 898 \end{array}$$

Step 5. Double the answer, 5, to obtain a trial divisor of 10. Divide the trial divisor into all but the last digit of the modified remainer. It will go into 89 eight times. Place the 8 above the second pair of digits, and also place the 8 to the right of the trail divisor. Thus, the true divisor is 108. Multiply 108 by 8 and obtain 864. Subtract 864 from 898 to obtain 34. Bring down the next pair of digits—89.

```
                        5  [8].
                      √33  98.89
                        25
2 × 5 = 10[8]            898
[8] × 108 =              864
                          3489
```

Note. With each new successive digit in the answer:

1. Place the digit in the answer above the pair of digits involved.
2. Place the same digit to the right of the trial divisor to obtain the true divisor.
3. Multiply the digit by the true divisor. (Do not use the square boxes in actual problems.)

Step 6. Double the answer, 58, to obtain a trial divisor of 116. Divide the trial divisor into all but the last digit of the remainder. It will go into 348 three times. Place the 3 above the third pair of digits, and also place the 3 to the right of the trial divisor. Thus, the true divisor is 1163. Multiply 1163 by 3 to obtain 3489. Subtract 3489 from 3489. There is no remainder. Therefore 3398.89 is a perfect square and its square root is 58.3.

```
                           5   8. [3]
                         √33  98.89
                           25
                            898
                            864
2 × 58 = 116 [3]             3489
[3] × 1163 =                 3489
```

Step 7. Check the answer by squaring 58.3—$58.3^2 = 3398.89$.

The complete calculation is shown below:

```
                               5   8. 3
                             √33  98.89
      5 × 5 =                  25
      2 × 5 = 10 [8]            898
   [8] × 108 =                  864
     2 × 58 = 116[3]             3489
  [3] × 1163 =                   3489
```

Example 2: Evaluate $\sqrt{786.808}$

Step 1. Starting at the decimal point, mark off the digits in pairs in both directions.

$\sqrt{07\ 86.80\ 80}$

Note. The extreme left-hand group may have only one digit. However, there must be an even number of digits to the right of the decimal point. If necessary, add a zero.

Step 2. Place the decimal point for the answer directly above the decimal point that appears under the radical sign.

$\sqrt{07\ 86.80\ 80}$

Step 3. Determine the largest number that can be squared without exceeding the first digit—7. The answer is 2, since the square of any whole number larger than 2 will be greater than 7. Place the 2 above the 7.

```
  2
√07  36.80  80
```

Step 4. Square 2 to obtain 4 and place it under 7. Subtract 4 from 7 to obtain 3. Bring down the next pair of digits—86.

```
  2
√07  86.80  80
  4
  386
```

Step 5. Double the answer, 2, to obtain a trial divisor of 4. Divide the trial divisor into all but the last digit of the modified remainder. It will go into 38 nine times. Place the 9 above the second pair of digits, and also place the 9 to the right of the trial divisor. The true divisor is 49. Multiply 49 by 9 to obtain 441. However, 441 cannot be subtracted from 386, so the next lower digit must be tried. Substitute 8 for 9 in both the answer and the divisor and multiply 48 by 8 to obtain 384. Subtract 384 from 386 to obtain a remainder of 2. Bring down the next pair of digits—80.

2 9.
$\sqrt{07\ 86.80\ 80}$
4
$2 \times 2 = 4$ [9] 386
[9] $\times 49 =$ 441

2 8.
$\sqrt{07\ 86.80\ 80}$
4
4 [8] 386
[8] $\times 48 =$ 384
280

Step 6. Double the answer, 28, to obtain a trial divisor of 56. Divide the trial divisor into all but the last digit of the remainder. Since it is not possible to divide 56 into 28, place a zero above the third pair of digits and bring down the next pair of digits—80.

2 8. [0]
$\sqrt{07\ 86.80\ 80}$
4
386
384
$2 \times 28 = 56$ 280

Step 7. Multiply 280 by 2 to obtain a trial divisor of 560. Divide the trial divisor into all but the last digit of the remainder. It will go 5 times. Place the 5 above the fourth pair of digits, and also place the 5 to the right of the trial divisor. Thus, the true divisor is 5605. Multiply 5605 by 5 to obtain 28025. Subtract 28025 from 28080. There is a remainder of 55. Thus, the square root of 786.808 is 28.05, with a remainder of 55. A more exact answer can be obtained by adding pairs of zeros and continuing the square root process.

2 8. 0 5
$\sqrt{07\ 86.80\ 80}$
4
386
384
$2 \times 280 = 560$ [5] 28080
[5] $\times 5605 =$ 28025
55

Check the answer by squaring 28.05 and adding the remainder ($28.05^2 + .0055$). Place the extreme right digit of the remainder under the extreme right digit of the squared number. The complete calculation is shown below:

2 8. 0 5
$\sqrt{07\ 86.80\ 80}$
$2 \times 2 = 4$ [8] 4
[8] $\times 48 =$ 386
$2 \times 28 = 56$ 384
$2 \times 280 = 560$[5] 28080
[5] $\times 5605 =$ 26025
55

25. Review Problems—Square Root

a. Solve the following:

(1) $\sqrt{441}$
(2) $\sqrt{1089}$
(3) $\sqrt{2500}$
(4) $\sqrt{8.40}$
(5) $\sqrt{2510.01}$
(6) $\sqrt{4901.4001}$
(7) $\sqrt{7482.25}$
(8) $\sqrt{5759.2921}$

b. Solve the following to nearest thousandth.

(1) $\sqrt{5}$
(2) $\sqrt{7}$
(3) $\sqrt{11}$
(4) $\sqrt{13}$
(5) $\sqrt{15}$
(6) $\sqrt{17}$

c. The current (in amperes) flowing through a resistor can be determined by taking the square root of the quotient obtained by dividing the value of power supplied to the resistor (in watts) by the value of the resistance (in ohms). Thus, if a resistance of 300 ohms is absorbing 60 watts of power, it is drawing a current of $\sqrt{\frac{60}{300}}$ amperes. This equals about .447 ampere. In the same manner, find the value of current for each of the following values of power and resistance:

	Power (watts)	*Resistance (ohms)*	*Current (amperes)*
(1)	25	1,000	?
(2)	50	7,000	?
(3)	40	500	?
(4)	75	60	?

CHAPTER 5
ALGEBRA

Section I. INTRODUCTION

26. General

a. Algebra is an extension of arithmetic. All of the four basic operations of arithmetic—addition, subtraction, multiplication and division—apply also to algebra. Arithmetic deals only with particular numbers; algebra may also employ letters or symbols to represent numbers.

b. Algebra is often referred to as the shorthand language of mathematicians. The simplest example of the algebraic language is the formula, in which letters are used to represent words or numbers. For example, the area (A) of a rectangle can be determined by multiplying the length (l) by the width (w). Algebraically, this is stated as $A = lw$.

27. Algebraic Expressions and Terms

a. An *algebraic expression* is the representation of any quantity in algebraic signs and symbols; for example, $2x - 7$. A *numerical algebraic expression* consists entirely of numerials and signs, such as $8 - (6 \times 2)$. A *literal algebraic expression* contains only letters and symbols, such as $ax - ay$.

b. Each algebraic expression contains two or more terms, separated by one of the signs of operation (+, —, ÷, ×). The expression $3x - 4xy - 2y$, for example, contains three terms: $3x$, $4xy$, and $2y$. If the terms have the same letters and exponents, such as $3a^2x$, $9a^2x$, and $12a^2x$, they are called *similar terms*. Terms that do not contain the same letters and exponents, such as $3ab^2$, $3a^2b$, and $3x^2y$, are *dissimilar terms*.

c. If an algebraic expression contains one term, such as $3abc$ or $5a^4x^2$, it is called a *monomial;* if it contains two terms, such as $x - y$, it is called a *binomial;* and if it contains three terms, such as $5x^2 - 3xy - 2y^2$, it is called a *trinomial*. A more general rule of algebraic expressions states that any expression containing more than one term is called a *polynomial.*

28. Signs of Operation

In algebra, the conventional signs of operation (+, —, × and ÷) retain the same meaning as in arithmetic. In algebra, however, certain other signs may be used.

a. Multiplication may be indicated as follows:

Arithmetic	*Algebra*
$a \times b$	ab
$a \times b$	$a\,.\,b$
$a \times b$	$(a)(b)$

b. Division may be indicated as follows:

Arithmetic	*Algebra*
$x \div y$	$\frac{x}{y}$
$(a + b) \div (a - b)$	$\frac{a+b}{a-b}$

c. The arithmetical signs for both *addition* and *subtraction* are retained in algebra.

Arithmetic	*Algebra*
$4 + 5$	$4 + 5$
$a - b$	$a - b$

29. Coefficients

Any factor of a product is known as a coefficient of the remaining factors. In the term $2\pi f$, 2 is the numerical coefficient of πf, f is the coefficient of 2π, and π is the coefficient of $2f$. However, it is common practice to speak of the numerical part of the term as the coefficient. If a term contains no numerical coefficient, the number 1 is understood. Thus, abc is 1 abc, and xyz is 1 xyz.

30. Subscripts

In expression such as $R_t = R_1 + R_2 + R_3$, the small numbers or letters written to the right and below the literal terms are called subscripts. Subscripts are used to designate different values of a variable quantity. They are read: *R* sub 1, *R* sub 2, etc.

31. The Radical Sign

The radical sign ($\sqrt{}$) has the same meaning in algebra as in arithmetic (ch. 5). Thus, the expression $z = 2\sqrt{R^2 + x^2}$ states that z is equal to 2 times the square root of $R^2 + x^2$.

Section II. POSITIVE AND NEGATIVE NUMBERS

32. Signed Numbers

Only positive numbers are used in arithmetical operations, but both *positive* and *negative* numbers may appear in algebraic expressions. The plus sign (+) is used to indicate a positive number and the minus sign (—) to indicate a negative number. If the sign is omitted, the number is understood to be positive. Positive and negative numbers are called *signed numbers.*

33. Need for Negative Numbers

The need for negative numbers may be seen from the succession of subtraction below:

6	6	6	6	6	6	6	6	6	6
—0	—1	—2	—3	—4	—5	—6	—7	—8	—9
6	5	4	3	2	1	0	—1	—2	—3

When the subtrahend is greater than the minuend, the difference becomes less than zero and the negative sign is placed before the difference. Thus, a negative number may be defined as a number less than zero.

34. Application of Positive and Negative Numbers

In technical work, many scales are calibrated above and below (or to the right and left of) a center point designated 0 (zero). For example, the degrees of temperature indicated on a thermometer scale are measurements of distance taken on a scale in opposite directions from some point chosen to represent a reference or zero point. Temperature is always so many degrees above or below zero. In mathematics, it is convenient to indicate that a temperature is so many degrees above or below zero by prefixing the reading with a positive or negative sign. Thus, 45° above zero is +45° and 15° below zero is —15°. Similarly, in electronic and electrical measuring instruments, scales are often calibrated to read positive numbers on one side of a zero and negative numbers on the other.

35. Graphical Representation of Positive and Negative Numbers

a. Principle. Positive and negative numbers may be represented graphically as shown in figure 4. The zero is the reference point. This graph can be used to illustrate both addition and subtraction.

b. Addition. To add numbers graphically, start at the zero reference point and mark off the first number, going to the right if the number is positive, or to the left if the number is

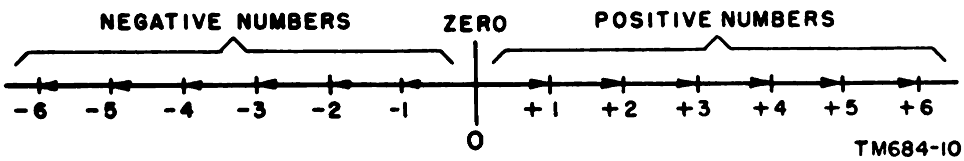


Figure 4. Graphical representation of positive and negative numbers.

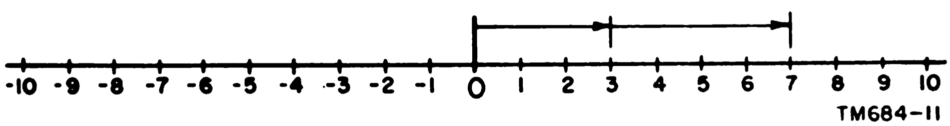


Figure 5. Graphical representation of addition of positive numbers.

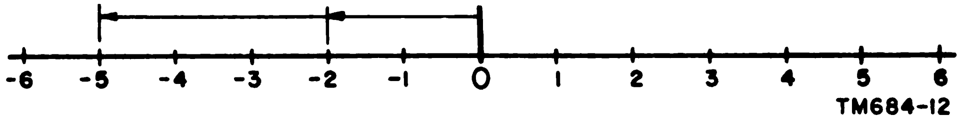


Figure 6. Graphical representation of addition of negative numbers (—1 and —5).

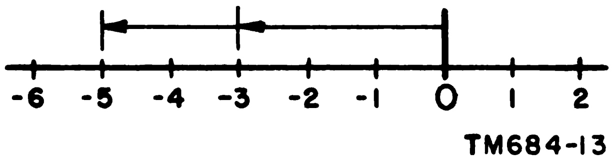


Figure 7. Graphical representation of addition of negative numbers (—3 and —2).

negative. From this new point, mark off the second number, again going to the right if the number is positive, or to the left if it is negative. The number of units between zero and the final point is the sum of the two numbers. This procedure can be continued for more than two numbers. Figure 5 shows graphical addition of positive numbers; figures 6 and 7 show graphical addition of negative numbers; and figure 8 shows the addition of a combination of a positive and a negative number. Figures 6 and 7 show that the order in which the negative numbers are taken does not affect the answer.

c. Subtraction. To subtract numbers graphically, change the sign of the subtrahend (number to be subtracted) and proceed as for addition. Figure 9 shows the subtraction of +3 from +5 to obtain the difference of +2.

36. Absolute Value of a Number

The numerical value of a number, without regard to its sign, is called the *absolute value* of the number. Thus, the absolute value of —3 or +3 is 3. This is written |3|.

37. Addition of Positive and Negative Numbers

a. Positive Numbers. To add two or more positive numbers, find the sum of their absolute values and prefix the sum with a plus sign. When there is no possibility of misunderstanding, the plus sign is usually omitted.

Example: Add +4, +5, and +6
$+4 + (+5) + (+6) = +15$ or 15

b. Negative Numbers. To add two or more negative numbers, find the sum of their absolute values and prefix the sum with a minus sign.

Example: Add —4, —5, and —6
$-4 + (-5) + (-6) = -15$

c. Positive and Negative Numbers. To add a positive and a negative number, find the difference between their absolute values and prefix the sum with the sign of the number that has the greater absolute value. This is called *algebraic addition.* When three or more positive and negative numbers are to be added, first find the sum of all positive numbers, and then the sum of all negative numbers. Add these sums algebraically as above.

Example 1: Add +6 and —9.
$+6 + (-9) = -3$

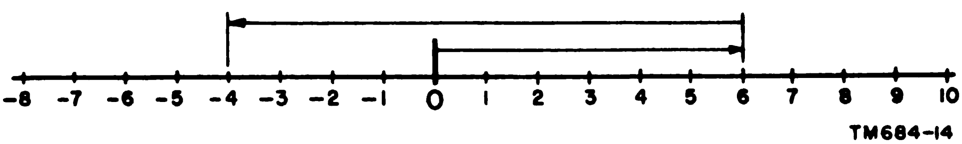


Figure 8. Graphical representation of addition of positive and negative numbers.

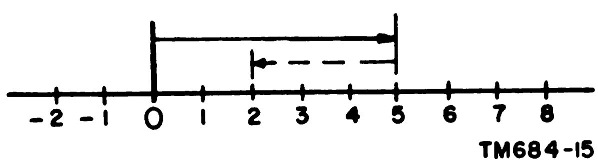


Figure 9. Graphical representation of subtraction of positive numbers.

Example 2: Add +5, —8, +12, and —6.

$+5 + (+12) = +17$
$-8 + (-6) = -14$
$(+17) + (-14) = +3$

38. Subtraction of Positive and Negative Numbers

To subtract positive and negative numbers, change the sign of the subtrahend and proceed as in addition (par. 37).

a. Positive Numbers.

Example 1: Subtract +2 from +5.
$+5 - (+2) = +5 - 2 = +3$ or 3

Example 2: Subtract $+5a^2$ from $+6a^2$.
$+6a^2 - (+5a^2) = +6a^2 - 5a^2 = +1a^2 = a^2$

b. Negative Numbers.

Example 1: Subtract —3 from —5.
$-5 - (-3) = -5 + 3 = -2$

Example 2: Subtract $-4a$ from $-2a$.
$-2a - (-4a) = -2a + 4a = +2a$ or $2a$

c. Positive and Negative Numbers.

Example 1: Subtract —2 from +5.
$+5 - (-2) = +5 + 2 = +7$ or 7.

Example 2: Subtract $-3x^2$ from $+5x^2$.
$+5x^2 - (-3x^2) = +5x^2 + 3x^2 = +8x^2$ or $8x^2$

39. Multiplication of Positive and Negative Numbers

a. Numbers Having Like Signs. If the two numbers to be multiplied have the same signs, the product is positive.

Example 1: Multiply +5 by +3.
$(+5)(+3) = +15$ or 15

Example 2: Multiply —5 by —3.
$(-5)\ (3-3) = +15$ or 15

b. Numbers Having Unlike Signs. If the two numbers to be multiplied have unlike signs, the product is negative.

Example 1: Multiply —5 by +3.
$(-5)(+3) = -15$

Example 2: Multiply +5 by —3.
$(+5)(-3) = -15$

c. Several Positive and Negative Numbers. To multiply several positive and negative numbers, multiply the numbers in groups of two in the order in which they appear.

Example 1:

Multiply $(-5)(+3)(+7)(-2)(-4)$.

$$
\begin{aligned}
& (-5)(+3) \quad (+7)(-2) \quad (-4) \\
= & \quad (-15) \qquad\quad (-14) \qquad (-4) \\
= & \qquad\quad (+210) \qquad\qquad (-4) \\
= & \qquad\qquad\qquad -840
\end{aligned}
$$

Example 2:

Multiply $(+7)(+2)(-5)(-3)(-1)(-4)$.

$$
\begin{aligned}
= & (+7)(+2) \quad (-5)(-3) \quad (-1)(-4) \\
= & \quad (+14) \qquad\quad (+15) \qquad\quad (+4) \\
= & \qquad\quad (+210) \qquad\qquad\quad (+4) \\
= & \qquad\qquad\qquad 840
\end{aligned}
$$

40. Division of Positive and Negative Numbers

a. Numbers Having Like Signs. The quotient of two numbers that have the same signs is positive.

Example 1: Divide —15 by —5.
$-15 \div -5 = +3$ or 3

Example 2: Divide +24 by +6.
$+24 \div +6 = +4$ or 4

b. Numbers Having Unlike Signs. The quotient of two numbers that have opposite signs is negative.

Example 1: Divide 35 by —7.
$+35 \div -7 = -5$

Example 2: Divide —8,988 by 28.
$-8988 \div 28 = -321$

41. Order of Signs

When only addition and subtraction signs appear in a series of terms, addition and subtraction procedures may be performed in any order. However, when multiplication and division signs appear in the same series with addition and subtraction signs, the multiplication and division must be performed first, and then the addition and subtraction.

Example 1: Evaluate $15 + 5 - 3 + 4 - 8$.

Step 1. Add the + terms:
$15 + 5 + 4 = 24$

Step 2. Add the — terms:
$(-3) + (-8) = -11$

Step 3. Add the + terms and — terms algebraically:
$24 - 11 = 13$.

Example 2: Evaluate $9 \times 4 + 6 - 3 + 5 \times 2$.

Step 1. Perform the multiplication first:
$(9 \times 4) + 6 - 3 + (5 \times 2) = (36) + 6 - 3 + (10)$

Step 2. Add the + terms:
$36 + 6 + 10 = 52$

Step 3. Add the + terms and the — terms algebraically:
$52 - 3 = 49$

Example 3: Evaluate $81 \div 9 - 3 + 6 - 15 + 4 \times 5$.

Step 1. Perform the division:
$(81 \div 9) - 3 + 6 - 15 + (4 \times 5) = (9) - 3 + 6 - 15 + (4 \times 5)$

Step 2. Perform the multiplication:
$9 - 3 + 6 - 15 + (4 \times 5) = 9 - 3 + 6 - 15 + (20)$

Step 3. Add the + terms:
$9 + 6 + 20 = 35$

Step 4. Add the — terms:
$(-3) + (-15) = -18$

Step 5. Add the ÷ terms and the — terms algebraically:
$35 - 18 = 17$.

42. Review Problems—Positive and Negative Numbers

a. Add the following:

(1) 23 and —6
(2) 21 and 37
(3) —54 and 33
(4) —43° and —96°
(5) 682 volts and —934 volts

b. Subtract the following:

(1) —104 amperes from 147 amperes
(2) —37 volts from —45 volts
(3) $.64cy$ from $.0025cy$
(4) $21.36ax^2$ from $-10.63ax^2$
(5) $-.986x^2y$ from $.824x^2y$

c. Find the product of the following:

(1) —6.4 and 2.8
(2) 3, —6, and 4
(3) $-\frac{2}{3}, -\frac{6}{7}$, and $-\frac{2}{5}$
(4) 3.01, —.02, and —1.26
(5) —.0025, 150, —.10, and .075
(6) —2, 5, 3, —1, and 4

d. Divide:

(1) 36 by 4
(2) $-\frac{5}{7}$ by $\frac{3}{4}$
(3) —5.6 by —.008
(4) —750 by —3
(5) $\frac{1}{3}$ ampere by $\frac{1}{2}$ ampere
(6) —.3750 by 150

e. Evaluate the following:

(1) $2 + 3 - 9$
(2) $3 + 4 + 2 \times 5 - 3$
(3) $2 - 3 \times 9$
(4) $3 \times 4 + 2 \times 5 - 3$
(5) $5 + 3 \times 7 - 2 \times 11 + 7$
(6) $28 \div 14 - 8 + 16 + 3 \times 2$
(7) $46 - 18 + 3 \times 4 - 8 + 12$
(8) $5 - 3 + 6 \times 4 + 40$
(9) $8 - 16 + 4 \times 3 - 10 \times 5$
(10) $15 \div 5 - 3 + 2 \times 10 - 2$

43. Addition and Subtraction of Algebraic Expressions

a. General. Only similar algebraic terms—those that are exactly alike in all respects other than numerical coefficients—may be added or subtracted. For example, the sum of $3x^2y$ and $5x^2y$ is $8x^2y$. Dissimilar terms cannot be added or subtracted directly, but the processes of addition or subtraction can be indicated by the use of plus or minus signs. For example, the sum of $4x^2y$ and $2xy^2$ is $4x^2y + 2xy^2$.

b. Procedure. To add or subtract algebraic expressions, arrange the terms so that like terms are in the same vertical column, and preferably in descending order of powers. Add or subtract the terms according to the rules of signed numbers (pars. 37 and 38).

Example 1: Add $x^3 - 3x^2 + 1$, $x^3 + x - 3$, and $x^2 + x + 1$.

$$\begin{array}{rrrr} x^3 & -\ 3x^2 & & +\ 1 \\ x^3 & & +\ x & -\ 3 \\ & x^2 & +\ x & +\ 1 \\ \hline 2x^3 & -\ 2x^2 & +\ 2x & -\ 1 \end{array}$$

Example 2: Subtract $x^3 + 3x^2 + x - 1$ from $x^4 + x^3 - x + 2$.

$$\begin{array}{rrrrr} x^4 & +\ x^3 & & -\ x & +\ 2 \\ & -(x^3 & -\ 3x^2 & -\ x & +\ 1) \end{array}$$

Remove parentheses and change signs.

$$\begin{array}{rrrrr} x^4 & +\ x^3 & & -\ x & +\ 2 \\ & -x^3 & -\ 3x^2 & -\ x & +\ 1 \\ \hline x^4 & & -\ 3x^2 & -\ 2x & +\ 3 \end{array}$$

44. Multiplication and Division of Monomials

a. Multiplication. In multiplying monomials, multiply the numerical coefficients and write this result as the coefficient of the product. After the coefficient, write each literal factor with an exponent equal to the sum of all the exponents of that letter in the original factors. For example, $3a^n \cdot 2a^m = 6a^{n+m}$.

Example 1: Multiply x^2 by x^3.

$x^2 \cdot x^3 = x^{2+3} = x^5$

Example 2: Multiply x, x^3, and x^{10}.

$x^1 \cdot x^3 \cdot x^{10} = x^{1+3+10} = x^{14}$

Example 3: Multiply x^3y^4 by $3xy^2$.

Step 1. Multiply the coefficients: $1 \cdot 3 = 3$

Step 2. Multiply the two factors having the base x: $x^3 \cdot x = x^{3+1} = x^4$

Step 3. Multiply the two factors having the base y: $y^4 \cdot y^2 = y^{4+2} = y^6$

Step 4. The product is: $x^3y^4 \cdot 3xy^2 = 3x^4y^6$

Example 4: Multiply x^2y^4z and wx^3yz^5.

$x^2y^4z \cdot wx^3yz^5 = wx^{2+3}y^{4+1}z^{1+5}$

$x^{2+3} = x^5$

$y^{4+1} = y^5$

$z^{1+5} = z^6$

Therefore, $x^2y^4z \cdot wx^3yz^5 = wx^5y^5z^6$.

b. Division. In dividing a monomial by a monomial, divide the numerical coefficient of the dividend by the coefficient of the divisor and write the result as the coefficient of the quotient. After the coefficient, write each literal factor with an exponent equal to its exponent in the dividend minus its exponent in the divisor. Thus, to divide $6a^n$ by $3a^m$ (*n* greater than *m*), $\frac{6a^n}{3a^m} = 2a^{n-m}$.

Example 1: Divide x^3 by x^2.

$\frac{x^3}{x^2} = x^{3-2} = x^1 = x$

Example 2: Divide $5x^6yz^3$ by $6x^3z^2$.

$$\frac{5x^6yz^3}{6x^3z^2} = \frac{5}{6}x^{6-3}yz^{3-2}$$

$$= \frac{5}{6}x^3yz \text{ or } \frac{5x^3yz}{6}$$

c. Removal of Parentheses and Brackets.

(1) In multiplying a quantity in parentheses by a given factor, multiply each term inside the parentheses by that factor and drop the parentheses. If the factor is a negative quantity, the sign of every term inside the parentheses is changed. For example, $-5(a - b + c) = -5a + 5b - 5c$.

(2) When an algebraic expression, such as $5x - 4\,[x - 2(x - 3)]$, has more than one grouping symbol (parentheses and brackets), remove the inside grouping symbol first and then successively remove the outer grouping symbols.

Example 1: Simplify $5x - 4\,[x - 2(x - 3)]$.

$$\begin{aligned} 5x - 4\,[x - 2(x - 3)] &= 5x - 4\,[x - 2x + 6] \\ &= 5x - 4x + 8x - 24 \\ &= 9x - 24 \\ &= 3(3x - 8) \end{aligned}$$

Example 2: Simplify $4a - \{6a - 2b + 2\,[2a - b + 42] - (c + 2b)\}$.

$$\begin{aligned} &4a - \{6a - 2b + 2\,[2a - b + 42] - (c + 2b)\}. \\ =\ &4a - \{6a - 2b + 4a - 2b + 84 - c - 2b\} \\ =\ &4a - 6a + 2b - 4a + 2b - 84 + c + 2b \\ =\ &-6a + 6b + c - 84 \end{aligned}$$

Example 3: Simplify $-\{-1\,[-(x - y - z) + 29] - 39 + 2y - z\}$.

$$\begin{aligned} &-\{-1\,[-(x - y - z) + 29] - 39 + 2y - z\} \\ =\ &-\{-1\,[-x + y + z + 29] - 39 + 2y - z\} \\ =\ &-\{+x - y - z - 29 - 39 + 2y - z\} \\ =\ &-x + y + z + 29 + 39 - 2y + z \\ =\ &-x - y + 2z + 68 \end{aligned}$$

45. Raising Algebraic Functions to Powers

To raise an algebraic function to a power, multiply the exponents. Thus, $(a^n)^m = a^{nm}$.

Example 1: Simplify $(5^3)^4$.

$$(5^3)^4 = 5^{3.4} = 5^{12}$$

Example 2: Simplify $(2ab)^3$.

$$(2ab)^3 = 2ab \cdot 2ab \cdot 2ab = 8a^3\,b^3$$
$$\text{or } 2^{1.3}a^{1.3}b^{1.3} = 8a^3b^3$$

Example 3: Simplify $(ax^2)^3$.

$$(ax^2)^3 = a^{1.3}x^{2.3} = a^3x^6$$

Example 4: Simplify $[(x^3)^4]^5$.

$$[(x^3)^4]^5 = [x^{3.4}]^5 = [x^{12}]^5 = x^{12.5} = x^{60}$$

Example 5: Simplify $\left(\frac{2}{x^2}\right)^5$

$$\left(\frac{2}{x^2}\right)^5 = \frac{2^{1.5}}{x^{2.5}} = \frac{2^5}{x^{10}} = \frac{32}{x^{10}}$$

46. Negative Exponents

The rule for dividing monomials (par. 44*b*) also holds when the exponents of the denominator is greater than the exponent of the numerator. For example, $a^3 \div a^5 = a^{3-5} = a^{-2}$; however, a quantity such as a^{-2} may be written as $\frac{1}{a^2}$.

Example: Multiply x^2, x^{-1}, and $\frac{1}{x^{-3}}$.

Step 1. Write down the factors of the multiplication:

$$x^2 \cdot x^{-1} \cdot \frac{1}{x^{-3}}$$

Step 2. Place all factors in the numerator:

$$x^2 \cdot x^{-1} \cdot x^3$$

Step 3. Multiply the factors (add their exponents):

$$x^{2-1+3} = x^4$$

47. Zero Exponents

The zero power of any quantity is equal to 1. For example $x^2 \cdot x^{-2} = x^0$ when the exponents are added. However, x^{-2} can also be written $\frac{1}{x^2}$; in this case, $x^2 \cdot x^{-2} = \frac{x^2}{x^2} = 1$. Therefore, $x^0 = 1$. ***Any number (except zero) raised to the zero power is equal to 1.***

Example: Solve $\dfrac{x^2y^3}{z} \cdot \dfrac{z^4}{xy} \div \dfrac{x^2y^2}{z^3}$.

$$\frac{x^2y^3}{z} \cdot \frac{z^4}{xy} \div \frac{x^2y^2}{z^3} = \frac{x^2y^3z^4}{xyz} \div \frac{x^2y^2}{z^3} = \frac{x^2y^3z^4}{xyz} \cdot \frac{z^3}{x^2y^2}$$

$$= \frac{x^2y^3z^7}{x^3y^3z} = x^{2-3}y^{3-3}z^{7-1}$$

$$= x^{-1}y^0z^6 = x^{-1} \cdot 1 \cdot z^6 = \frac{z^6}{x}$$

48. Multiplication of Polynomials

a. ***By a Monomial.*** *To multiply a polynomial by a monomial, multiply each term in the polynomial separately by the monomial and add the products.* Observe the rules for the multiplication of signed numbers (par. 39) and exponents (par. 44a).

Example 1: Multiply $3a + 2ab + 5c$ by $2b$.

$$\begin{array}{r} 3a + 2ab + 5c \\ 2b \\ \hline 6ab + 4ab^2 + 10bc \end{array}$$

Example 2: Multiply $ad - ae + af$ by $3a^2$.

$$\begin{array}{r} ad - ae + af \\ 3a^2 \\ \hline 3a^3d - 3a^3e + 3a^3f \end{array}$$

Example 3: Multiply $3x^2y^2 - 2xy^3 + 5x^4y$ by $4x^3y$.

$$\begin{array}{r} 3x^2y^2 - 2xy^3 + 5x^4y \\ 4x^3y \\ \hline 12x^5y^3 - 8x^4y^4 + 20x^7y^2 \end{array}$$

b. ***By a Polynomial.*** *To multiply a polynomial by another polynomial, multiply each term of one polynomial by each term of the other and add the products.*

Example 1: Multiply $(a + b)$ by $(a + b)$.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

Example 2: Multiply $2x + 3y$ by $2x + 3z$.

$$\begin{array}{r} 2x + 3y \\ 2x + 3z \\ \hline 4x^2 + 6xy \\ + 6xz + 9yz \\ \hline 4x^2 + 6xy + 6xz + 9yz \end{array}$$

Example 3: Multiply $5x^2 - 6xy + 3y^2$ by $x + y$.

$$\begin{array}{r} 5x^2 - 6xy + 3y^2 \\ x + y \\ \hline 5x^3 - 6x^2y + 3xy^2 \\ + 5x^2y - 6xy^2 + 3y^3 \\ \hline 5x^3 - x^2y - 3xy^2 + 3y^3 \end{array}$$

49. Division of Polynomials

a. ***By a Monomial.*** *To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.*

Example 1: Divide $3a^2 + 4ab + 5ac$ by a.

$$\frac{3a^2 + 4ab + 5ac}{a} = 3a + 4b + 5c$$

Example 2: Divide $7x^2 + 14xy - 21ax^2$ by $7x$.

$$\frac{7x^2 + 14xy - 21ax^2}{7x} = x + 2y - 3ax$$

Example 3: Divide $4r(s + t) - r^3(s + t)^2 + qr^2(s + t)^3$ by $r^2(s + t)$.

$$\frac{4r(s + t) - r^3(s + t)^2 + qr^2(s + t)^3}{r^2(s + t)}$$

$$= \frac{4r(s + t)}{r^2(s + t)} - \frac{r^3(s + t)^2}{r^2(s + t)} + \frac{qr^2(s + t)^3}{r^2(s + t)}$$

$$= \frac{4}{r} - r(s + t) + q(s + t)^2$$

b. By a Polynomial. To divide a polynomial by a polynomial, just arrange the dividend and the divisor according to descending powers of one variable, starting with the highest powers at the left. Then proceed as shown in the examples below. **If there is a remainder, write it as the numerator of a fraction the denominator of which is the divisor.**

Example 1: Divide $ab + ac + db + dc$ by $a + d$.

Step 1. Divide the first term of the divisor, a, into the first term of the dividend, ab. The quantity a is contained in the first term, ab, b times. Write b as the first term of the quotient.

$$\begin{array}{r|l} & b \\ \hline a+d & ab + ac + db + dc \end{array}$$

Step 2. Multiply both terms of the divisor by b:

$$\begin{array}{r|l} & b \\ \hline a+d & ab + ac + db + dc \\ & ab \qquad + db \\ \hline \end{array}$$

Step 3. Subtract the result from the original dividend:

$$\begin{array}{r|l} & b \\ \hline a+d & ab + ac + db + dc \\ & ab \qquad + db \\ \hline & \qquad ac \qquad + dc \end{array}$$

Step 4. Divide the first term of the divisor into the first term of the remainder. It is contained in the first term, ac, c times. Write c as the second term of the quotient.

$$\begin{array}{r|l} & b + c \\ \hline a+d & ab + ac + db + dc \\ & ab \qquad + db \\ \hline & \qquad ac \qquad + dc \end{array}$$

Step 5. Multiply both terms of the divisor by c and subtract. There is no remainder:

$$\begin{array}{r|l} & b + c \\ \hline a+d & ab + ac + db + dc \\ & ab \qquad + db \\ \hline & \qquad ac \qquad + dc \\ & \qquad ac \qquad + dc \\ \hline \end{array}$$

Step 6. Therefore,

$$\frac{ab + ac + db + dc}{a + d} = b + c.$$

Example 2: Divide $x^2 + 2xy + y^2$ by $x + y$.

$$\frac{x^2 + 2xy + y^2}{x + y} =$$

$$\begin{array}{r|l} & x + y \\ \hline x+y & x^2 + 2xy + y^2 \\ & x^2 + \ \ xy \\ \hline & \qquad xy + y^2 \\ & \qquad xy + y^2 \\ \hline \end{array}$$

Therefore,

$$\frac{x^2 + 2xy + y^2}{x + y} = x + y.$$

Example 3: Divide $6a - ab - 27ac - 15b^2 + 7bc + 30c^2$ by $3a - 5b - 6c$.

$$\frac{6a^2 - ab - 27ac - 15b^2 + 7bc + 30c^2}{3a - 5b - 6c} =$$

$$\begin{array}{r|l} & 2a + 3b - 5c \\ \hline 3a - 5b - 6c & 6a^2 - ab - 27ac - 15b^2 + 7bc + 30c^2 \\ & 6a^2 - 10ab - 12ac \\ \hline & \qquad 9ab - 15ac - 15b^2 + 7bc + 30c^2 \\ & \qquad 9ab \qquad\quad - 15b^2 - 18bc \\ \hline & \qquad\quad - 15ac \qquad\quad + 25\,bc + 30c^2 \\ & \qquad\quad - 15ac \qquad\quad + 25\,bc + 30c^2 \\ \hline \end{array}$$

50. Review Problems—Fundamental Operations

a. Add the following algebraic expressions:

(1) $2a^4 + 3a^2b^2 + 5b^4$, $a^4 - 5a^2b^2 - 2b^4$, and $3a^4 - 2a^2b^2 + b^4$.

(2) $3E - 2RI - 15ZI$, $6RI + 24ZI$, and $-2E - RI + 11ZI$.

(3) $10w - 4x + 3y + 6z$, $2x - 5w + y$, $3z - 2x - y$, and $6y - 4w - z + 5x$.

b. Subtract the following algebraic expressions:

(1) $-7ax - 2by + cz$ from $12ax + 15by - 8cz$.

(2) $10w - 3y - 4z + 6x$ from $3x + 5y - 2z - 15w$.

(3) $8a^2 + 10ab - 4b^2$ from $12a^2 - 24ab + 2b^2$.

c. Simplify:

(1) $7a^0$

(2) $(5x + 9)^0$

(3) $(3x^2 + 7x + 1)^0$

d. Perform the indicated operations:

(1) $f^6 \cdot f^4$

(2) $y^a \cdot y^b$

(3) $v^{x+1} \cdot v^{x-1}$

(4) $\frac{r^{10}}{r^5}$

(5) $(R^3)^m$

(6) $\frac{r^{m+5}}{r^4}$

e. Express with positive exponents:

(1) $4x^{-4}$

(2) $r^{-3}x^{-4}$

(3) $(6a)^{-2b}$

(4) $I^{-2}R^{-1}$

(5) $2^{-3}a^2b^{-3}$

(6) $\frac{3EI^{-2}R^{-1}}{4}$

f. Perform the indicated operations:

(1) $(5ab)(2a^2 - 3ab + 7b^2)$

(2) $4a(a^2 + 3a + 1)$

(3) $(i^2 + 3i + 9)(i - 3)$

(4) $(2x^2 + 3xy - y^2)(x^2 + xy + y^2)$

(5) $(3x^2 - 2xy - 5y^2)(3x^2 + 2xy - 5y^2)$

(6) $[(x - 1)a - (x - 1)c] \div [(x - 1)\,ac]$

(7) $(3rL - rR^2) \div rR$

(8) $(5a^4b - 10a^6b^2 + 15a^3b^4) \div 5a^4b$

(9) $(1 + 2z^4 + 4z^2 - z^3 + 7z) \div (3 + z^2 - z)$

(10) $(100b^3 - 13b^2 - 3b) \div (3 + 25b)$

Section IV. FACTORING

51. Understanding Factoring

Factoring is the breaking up of an expression into the *factors* or *individual parts* of which it is composed. In other words, to factor an algebraic expression means to find two or more expressions which, when multiplied together, will result in the original expression. For example, since $3 \cdot 5 = 15$, 3 and 5 are the factors of 15; since $4 \cdot a \cdot b = 4ab$, 4, a, and b, are the factors of $4ab$; since $a(x + y) = ax + ay$, a and $(x + y)$ are the factors of $ax + ay$.

52. Factors of Positive Integers

It is often difficult to determine at a glance the factors of which a number is composed. For example, consider the numerical expression 36. There are many different combinations of numbers that would result in an answer of 36; for example, the desired factors for 36 in a certain problem might $36 \cdot 1$, $18 \cdot 2$, $12 \cdot 3$, $9 \cdot 4$, $6 \cdot 6$, $2 \cdot 2 \cdot 9$, $4 \cdot 3 \cdot 3$, $2 \cdot 3 \cdot 6$, and so on.

53. Factors of a Monomial

Because the factors of a monomial are evident, usually a monomial is not separated into its prime factors. The factors of a^4b^2c are $a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot c$, and the factors of $15a^2b^3$ are $3 \cdot 5 \cdot a \cdot a \cdot b \cdot b \cdot b$.

54. Square Root of a Monomial

The square root of an algebraic expression is one of its two equal factors. Thus, the square root of 49 is 7, the square root of 81 is 9, the square root of a^2 is a, and the square root of x^2y^2 is xy. As discussed in paragraph 31, the radical sign is used to indicate the square root of a number. Actually, every number has two square roots, one positive and one negative. If no sign precedes the radical, the positive or *principal root* is understood. For example, $\sqrt{9} = +3$. If a negative sign *precedes the radical*, however, the negative root is intended. Thus, $-\sqrt{9} = -3$. When dealing with literal terms, the values of the various factors often are unknown. Therefore, *when extracting the square root of a monomial, extract the square root of the numerical coefficient, divide the exponents of the literal terms by 2, and prefix the square root with the plus or minus (±) sign,* which denotes that either the positive or negative root may be the correct one.

Example 1: $\sqrt{x^{16}y^4} = \pm x^8y^2$.

Example 2: $\sqrt{49a^4b^2} = \pm 7a^2b$.

55. Cube Root of a Monomial

The cube root of a monomial is one of its three equal factors. The index 3 in the angle of the radical sign ($\sqrt[3]{\ }$) indicates cube root (par. 31). *To extract the cube root of a monomial, extract the cube root of the numerical coefficient, divide the exponents of the literal terms by 3, and prefix the cube with the same sign as that of the monomial.*

Example 1: $\sqrt[3]{a^6y^3} = a^2y$.

Example 2: $\sqrt[3]{27x^{12}y^6z^9} = 3x^4y^2z^3$.

Example 3: $\sqrt[3]{-64r^{21}s^3} = -4r^7s$.

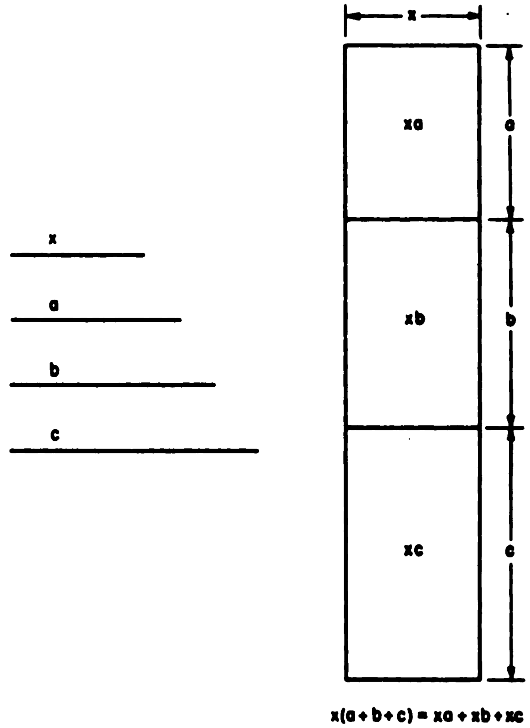


Figure 10. Common monomial factors.

56. Factors of a Polynomial

a. Common Monomial Factor. In an algebraic expression, the type of factor which can be recognized most easily is the monomial factor (single letter or number) which is common to each term in the expression. For instance, in the expression $xa + xb + xc$, the x is a factor common to each of the terms. Thus, the expression $xa + xb + xc$ can be written $x(a + b + c)$. This relationship is shown pictorially in figure 10. Since the area of a rectangle is equal to its base multiplied by its altitude (par. 136*b*), the area of the uppermost rectangle in figure 10 is x times a, or xa. The areas of the center and lower rectangles are xb and xc, respectively. The area of the large rectangle formed by the three small rectangles is equal to its base x times its altitude $(a + b + c)$, or $x(a + b + c)$. Since the area of the large rectangle is equal to the sum of the areas of the three smaller rectangles, then $x(a + b + c)$ is equal to $xa + xb + xc$. This shows that the factor x can be removed from $xa + xb + xc$ and the expression written $x(a + b + c)$. Accuracy of factoring can be checked by multiplying the two factors together—the product should be the original expression. Thus, $x(a + b + c) = xa + xb + xc$. *To factor a polynomial the terms of which have a common monomial factor, determine the largest factor common to all of the terms, divide the polynomial by this factor, and write the quotient in parentheses preceded by the monomial factor.* The first factor contains all that is common to all of the terms; it may consist of more than one literal number and may be to a power higher than the first.

Example 1: Factor $x^3 - 7x^2 + 4x$.
$x^3 - 7x^2 + 4x = x(x^2 - 7x + 4)$

Example 2: Factor $abx + aby - abz$.
$abx + aby - abz = ab(x + y - z)$

Example 3: Factor $2az^2 - 4bz^2 + 6cz^2$.
$2az^2 - 4bz^2 + 6cz^2 = 2z^2(a - 2^b + 3^c)$

b. Binomial Factors. Sometimes binomial factors are not immediately apparent, and an algebraic term may appear to have no common factors. For example, the expression $am + bm + an + bn$ may seem to have no factors in common. However, the first pair, $am + bm$, has a common factor, m, and the second pair, $an + bn$, has a common factor, n. Factoring out the common factors, the expression becomes $m(a + b) + n(a + b)$. Since there are two terms containing a common factor $(a + b)$, this factor can be removed to make the expression $(a + b)(m + n)$. Thus, the factors are $(a + b)$ and $(m + n)$. This relationship is shown pictorially in figure 11. Starting with the upper left-hand rectangle and going clockwise, the areas of the four rectangles are an, am, bm, and bn. The area of the large rectangle formed by the four smaller rectangles is its base $(m + n)$ times its altitude $(a + b)$, or $(m + n)(a + b)$. Since the area of the large rectangle is equal to the sum of the areas of the four smaller rectangles, then $(m + n)(a + b)$ is equal to $an + am + bm + bn$. This shows that the expression $am + bm + an + bn$ can be factored into $(m + n)$ and $(a + b)$. To check the factoring, multiply $(a + b)$ by $(m + n)$; the product is $am + an + bm + bn$. Since the addition of terms can be expressed in any order, the factoring is correct.

Example 1: Factor $py - pz - qy + qz$.

$$py - pz - qy + qz = p(y - z) - q(y - z)$$
$$= (p - q)(y - z)$$

Example 2: Factor $4xa - 8zb - 6ya - 4xb + 8za + 6yb$.

$$4xa - 8zb - 6ya - 4xb + 8za + 6yb$$
$$= 4xa - 6ya + 8za - 4xb + 6yb - 8zb$$
$$= 2a(2x - 3y + 4z) - 2b(2x - 3y + 4z)$$
$$= (2a - 2b)(2x - 3y + 4z)$$
$$= 2(a - b)(2x - 3y + 4z)$$

Example 3: Factor $da + db - dc - ea - eb + ec + fa + fb - fc$.

$$da + db - dc - ea - eb + ec + fa + fb - fc$$
$$= d(a + b - c) - e(a + b - c) + f(a + b - c)$$
$$= (d - e + f)(a + b - c)$$

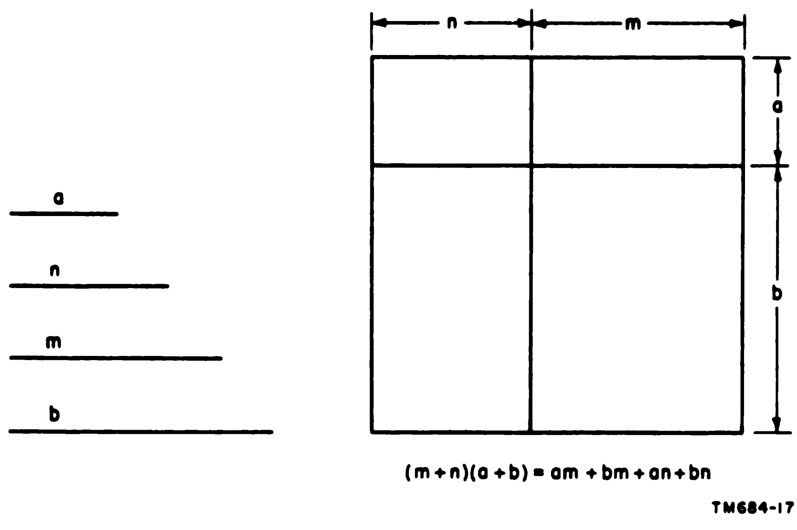


Figure 11. Binomial factors.

57. Factors of the Square of a Binomial

a. Square of Sum of Two Numbers. The square of the sum of two numbers is a special product that should be readily recognized to aid in factoring algebraic expressions. *The square of the sum of two numbers equals the square of the first, plus twice the product of the first and second, plus the square of the second.* To illustrate, $(a + b)^2 = a^2 + 2ab + b^2$. Conversely, the factors of $a^2 + 2ab + b^2$ are $(a + b)$ $(a + b)$ or $(a + b)^2$. This relationship is shown in figure 12. The areas of the four rectangles, as shown on the figure, are a^2, ab, ab, and b^2. The area of the large rectangle formed by the four smaller rectangles is equal to its base (a + b) times its altitude $(a + b)$, or $(a + b)^2$. Since the area of the large rectangle is equal to the sum of the areas of the four smaller rectangles, then $(a + b)^2$ is equal to $a^2 + ab + ab + b^2$, or $a^2 + 2ab + b^2$. This shows that the expression $a^2 + 2ab + b^2$ can be factored into $(a + b)$ $(a + b)$, or $(a + b)^2$. Figure 13 shows a similar relationship in which nine small rectangles form one large rectangle. In this case, the area of the large rectangle is $(a + 2b)^2$ and the sum of the areas of the nine smaller rectangles is $a^2 + 4ab + 4b^2$; consequently, $(a + 2b)$ and $(a + 2b)$ are factors of $a^2 + 4ab + 4b^2$. Thus, the factors of the square of one number, plus twice the product of the first and second number, plus the square of the second number are the square of the sum of the two numbers.

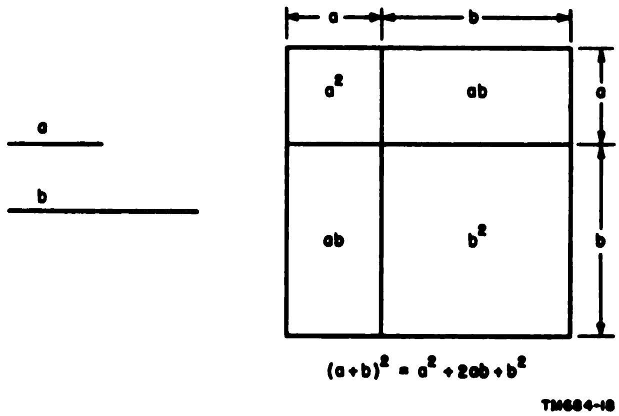


Figure 12. Square of sum of two numbers.

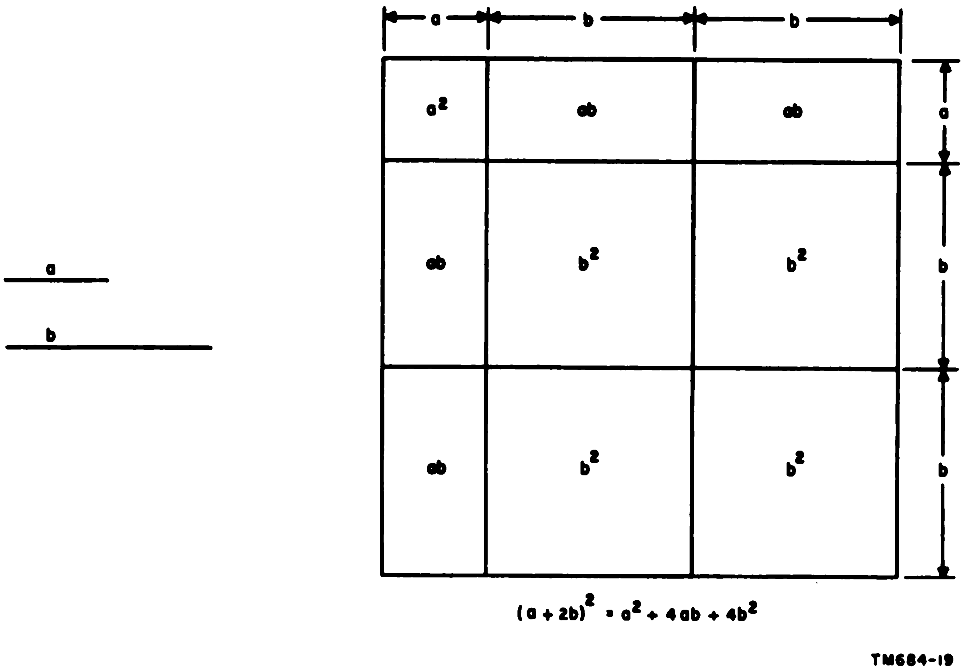


Figure 13. Factors of square of positive binomial.

Example: Factor $4b^2 + 16db + 16d^2$.

$$4b^2 + 16db + 16d^2 = (2b + 4d)\ (2b + 4d)$$
$$= (2b + 4d)^2$$
$$= [2(b + 2d)]^2$$
$$= 2^2\ (b + 2d)^2$$

To prove the factoring:

$$(2b + 4d)^2 = (2b)^2 + 2(2b)\ (4d) + (4d)^2$$
$$= 4b^2 + 16db + 16d^2$$

Note that 4 (that is, 2^2) may be removed before factoring the rest of the expression—this often simplifies computation.

$$4(b^2 + 4bd + 4d^2) = 4(b + 2d)^2$$

b. Square of Difference of Two Numbers. The square of the difference of two numbers equals the square of the first, minus twice the product of the first and second, plus the square of the second. For example, $(a - b)^2 = a^2 - 2ab + b^2$. The factors of $a^2 - 2ab + b^2$ are $(a - b)\ (a - b)$ or $(a - b)^2$. This relationship is shown pictorially in figure 14. The area of the large rectangle formed by the four small rectangles is a^2. The areas of the four smaller rectangles are shown on the illustration. The area of the upper left-hand rectangle is $(a - b)^2$. It is also equal to the area of the large rectangle minus the areas of the other three rectangles, or $a^2 - b\ (a - b) - b\ (a - b) - b^2$. This can be further simplified as follows:

$a^2 - b\ (a - b) - b\ (a - b) - b^2$
$a^2 - 2b\ (a - b) - b^2$
$a^2 - 2ab + 2b^2 - b^2$
$a^2 - 2ab + b^2$

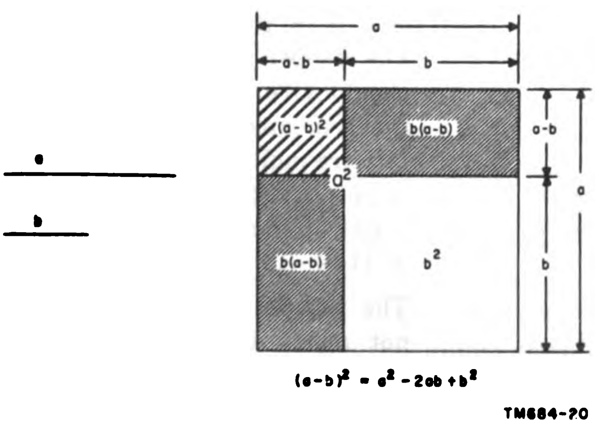


Figure 14. Square of difference of two numbers.

Therefore, $(a - b)^2 = a^2 - 2ab + b^2$, and $(a - b)$ and $(a - b)$ are factors of $a^2 - 2ab + b^2$. Thus, the factors of the square of one number, minus twice the product of the first and the second, plus the square of the second are the square of the difference of the two numbers.

Example:

Factor $9b^2 - 12bd + 4d^2$.

$$9b^2 - 12bd + 4d^2 = (3b - 2d)\ (3b - 2d)$$
$$= (3b - 2d)^2$$

To prove the factoring:

$$(3b - 2d)^2 = (3b)^2 - 2\ (3b)\ (2d) + (2d)^2$$
$$= 9b^2 - 12bd + 4d^2$$

58. Factors of Difference of Two Squares

The product of the sum and difference of two numbers is equal to the difference of their squares. Thus, $(a + b)\ (a - b) = a^2 - b^2$. To factor the difference of two squares, extract the square roots, then write the sum of the roots as one factor and the difference of the roots as the other factor. Thus, the factors of $a^2 - b^2$ are $(a + b)\ (a - b)$.

Example:

Factor $4x^2 - 9y^2$.

$$4x^2 - 9y^2 = (2x + 3y)\ (2x - 3y)$$

To prove the factoring:

$$(2x + 3y)\ (2x - 3y)$$
$$= (2x)^2 + (2x)(3y) - (2x)(3y) - (3y)^2$$
$$= 4x^2 - 9y^2$$

59. Factors of Trinomials

a. Trinomials Such as $x^2 + x(a + b) + ab$. The factors of a trinomial consisting of the square of the common term, the product of the common term and the algebraic sum of the unlike terms, and the product of the unlike terms are two binomials that have one term in common and the other term unlike. Thus, the factors of $x^2 + x(a + b) + ab$ are $(x + a)(x + b)$ where x is the common term, and a and b are the unlike terms. As proof, the product of $(x + a)(x + b)$ is $x^2 + xa + xb + xb + ab$. By factoring the two terms which have a common factor, x, the original trinomial $x^2 + x(a + b) + ab$ is obtained.

Example: Factor $9r^2 + 6r(s + t) + 4st$.

$$9r^2 + 6r(s + t) + 4st = (3r + 2s)(3r + 2t)$$

To prove the factoring:

$$(3r + 2s)(3r + 2t) = (3r)^2 + (3r)(2s) + (3r)(2t) + (2s)(2t)$$
$$= 9r^2 + 6rs + 6rt + 4st$$
$$= 9r^2 + 6r(s + t) + 4st$$

b. Trinomials Such as $x^2 + 6x + 8$. To factor a trinomial of the form $x^2 + 6x + 8$, $x^2 - 6x + 8$, $x^2 + 6x - 8$, or $x^2 - 6x - 8$, much of the work is done by trial and error. The problem is to find two factors of the final term which, when added together, will give the coefficient of the middle term. Taking the first of the trinomials above, the factors of 8 are $8 \cdot 1$ and $4 \cdot 2$. Since $4 + 2 = 6$ and $8 + 1 = 9$, the factors that will be used are 4 and 2. With regards to signs, *if the sign of the final term is positive, the signs of the two factors are alike and will be the same as the sign of the middle term.* Thus, the factors $x^2 + 6x + 8$ are $(x + 4)$ and $(x + 2)$, and the factors of $x^2 - 6x + 8$ are $(x - 4)$ and $(x - 2)$. *If the sign of the final term is negative,* however, *the signs containing the two terms of each binomial factor are unlike; the larger factor will take the sign of the middle term.* For example, the factors of $x^2 + 2x - 8$ are $(x + 4)$ and $(x - 2)$, and the factors of $x^2 - 2x - 8$ are $(x - 4)$ and $(x + 2)$.

Example 1: Factor $y^2 + 12y + 32$.
$$y^2 + 12y + 32 = (y + 8)(y + 4)$$

Example 2: Factor $z^2 - 11z + 30$.
$$z^2 - 11z + 30 = (z - 6)(z - 5)$$

Example 3: Factor $r^2 + 4r - 12$.
$$r^2 + 4r - 12 = (r + 6)(r - 2)$$

Example 4: Factor $s^2 - s - 20$.
$$s^2 - s - 20 = (s - 5)(s + 4)$$

c. Trinomials Such as $6a^2 - 11a - 10$. The procedure used to factor trinomials of this type is an extension of the procedure described in *b* above and as shown in the example below.

Example: Factor $6a^2 - 11a - 10$.

Step 1. Find two numbers that, when multiplied together, form the left-hand term, $6a^2$.

$$(6a)(a) = 6a^2$$
$$(2a)(3a) = 6a^2$$

Step 2 Find two numbers that, when multiplied together, form the right-hand term, -10.

$$(10)(-1) = -10$$
$$(5)(-2) = -10$$
$$(-10)(1) = -10$$
$$(-5)(2) = -10$$

Step 3. By trial and error, set up two binomial expressions containing factors from step 1 in the left-hand term and factors from step 2 in the right-hand term. The proper selection of factors should give the middle term of the trinomial when the binomials are multiplied.

$(2a + 5)(3a - 2)$ (first trial)
$6a^2 + 15a - 4a - 10 = 6a^2 + 11a - 10$ (multiplying out)

The middle term obtained does not match the middle term of the given trinomial. The numerical value is correct, but the sign is wrong. Make a second trial with the signs in the binomials changed.

$(2a - 5)(3a + 2)$

$6a^2 - 15a + 4a - 10 = 6a^2 - 11a - 10$

Step 4. Since the second trial results in the correct trinomial, the factors of $6a^2 - 11a - 10$ are $(2a - 5)$ and $(3a + 2)$.

Note. The method of trial and error used above may not work in every case. Other arrangements of factors and signs must be tried until the correct results are obtained.

60. Factors of Two Cubes

a. Sum of Two Cubes. The factors of the sum of two cubes, such as $x^3 + y^3$, are $(x + y)$ and $(x^2 - xy + y^2)$. In this case, the binomial is an expression of the sum of the primes times the sum of the squares of the primes minus the product of the primes. This is seen readily by dividing $x^3 + y^3$ by $x + y$.

Thus,

$$\begin{array}{r l} & x^2 - xy + y^2 \\ x + y\,/ & x^3 + y^3 \\ & x^3 + x^2y \\ & -x^2y \\ & -x^2y - xy^2 \\ & xy^2 + y^3 \\ & xy^2 + y^3 \end{array}$$

Example 1: Factor $z^3 + 8$.

$z^3 + 8 = (z + 2)(z^2 - 2z + 4)$

To prove the factoring:

$$\begin{array}{r l} & z^2 - 2z + 4 \\ z + 2\,/ & z^3 + 8 \\ & z^3 + 2z^2 \\ & -2z^2 \\ & -2z^2 - 4z \\ & 4z + 8 \\ & 4z + 8 \end{array}$$

Example 2:

Factor $r^3 + 125x^3$.

$r^3 + 125x^3 = (r + 5x)(r^2 - 5rx + 25x^2)$

To prove the factoring:

$$\begin{array}{r l} & r^2 - 5rx + 25x^2 \\ r + 5x\,/ & r^3 + 125x^3 \\ & r^3 + 5r^2x \\ & -5r^2x \\ & -5r^2x - 25rx^2 \\ & 25rx^2 + 125x^3 \\ & 25rx^2 + 125x^3 \end{array}$$

b. Difference of Two Cubes. The factors of the difference of two cubes, such as $x^3 - y^3$, are $(x - y)(x^2 + xy + y^2)$. These factors are an expression of the difference of the primes times the sum of the squares plus the product of the primes. As in the sum of two cubes, factoring can be proved by dividing the product by the binomial factor.

Example 1: Factor $a^3 - b^3$.

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

To prove the factoring:

$$\begin{array}{r l} & a^2 + ab + b^2 \\ a - b\,/ & a^3 - b^3 \\ & a^3 - a^2b \\ & a^2b \\ & a^2b - ab^2 \\ & ab^2 - b^3 \\ & ab^2 - b^3 \end{array}$$

Example 2: Factor $z^3 - 27$.

$z^3 - 27 = (z - 3)(z^2 + 3z + 9)$

To prove the factoring:

$$\begin{array}{r l} & z^2 + 3z + 9 \\ z - 3\,/ & z^3 - 27 \\ & z^3 - 3z^2 \\ & 3z^2 \\ & 3z^2 - 9z \\ & 9z - 27 \\ & 9z - 27 \end{array}$$

Example 3: Factor $64s^3 - 216t^3$.

$$64s^3 - 216t^3 = (4s - 6t)(16s^2 + 24st + 36t^2)$$

To prove the factoring:

$$\begin{array}{r|l}
 & 16s^2 \quad + 24st \quad + 36t^2 \\
4s - 6t & 64s^3 \qquad\qquad\qquad - 216t^3 \\
 & \underline{64s^3 - 96s^2t} \\
 & \qquad 96s^2t \\
 & \qquad \underline{96s^2t - 144st^2} \\
 & \qquad\qquad 144st^2 - 216t^3 \\
 & \qquad\qquad \underline{144st^2 - 216t^3}
\end{array}$$

61. Review Problems—Factoring

a. Factor:

(1) $25 + 5 - 30$

(2) $8 + 4 - 32$

(3) $9 - 18 + 21$

(4) $7r - 21r + 35r$

(5) $10x + 8y + 6z$

b. Find the values of the indicated powers:

(1) $(7xy^3)^2$

(2) $(-2w^5)^2$

(3) $(8a^2b^4)^2$

(4) $(9a^3x)^3$

(5) $(-3bz^4)^3$

c. Find the value of each of the following:

(1) $\sqrt{5^2}$

(2) $\sqrt{4^3}$

(3) $\sqrt{a^2b^6}$

(4) $\sqrt{36y^2z^4}$

(5) $\sqrt{100a^2b^{10}}$

(6) $\sqrt{16a^2 \cdot 5^2}$

(7) $\sqrt[3]{-27}$

(8) $\sqrt[3]{-x9}$

(9) $\sqrt[3]{(-8)^2}$

(10) $\sqrt[3]{125x^{12}y^{15}z^6}$

d. Factor:

(1) $3x + 6$

(2) $5a^2 + 15a$

(3) $10x^3 - 14x^2 - 2x$

(4) $6azy + 9bzx - 12cz$

(5) $m^3 + m^2 - 5mx$

(6) $3a^5 - 6a^4b - 3a^3b^2$

(7) $7ry^3 - 14ry^3 + 21ry^3$

(8) $12x^2am + 14xa^2m + 16xam^2$

(9) $\pi r\frac{2}{1} + \pi r\frac{2}{2}$

(10) $\frac{1}{4}c^3d - \frac{1}{8}c^2d^2 + \frac{1}{16}cd^3$

Section V. ALGEBRAIC FRACTIONS

62. General

Algebraic fractions play an important part in equations for electrical and electronic circuits. These fractions can be added, subtracted, multiplied, and divided in the same manner as arithmetical fractions.

63. Changing Signs of Fractions

a. The sign preceding a fraction is the sign of the fraction. It refers to the fraction as a whole and not to either the numerator or the denominator. In addition, the numerator and denominator each has a sign. For example, in the fraction $-\frac{3a}{5b}$, the sign of the fraction is minus, the sign of the numerator is plus, and the sign of the denominator is plus. Any two of the three signs can be changed without changing the value of the fraction.

$$\text{Thus, } -\frac{3a}{5b} = \frac{-3a}{5b} = \frac{3a}{-5b}.$$

Therefore, the sign of the fraction is not changed if the signs of both the numerator and the denominator are changed. Also, the sign of the fraction must be changed if the sign of either the numerator or denominator, but not both, is changed.

b. If the numerator or denominator is a polynomial, the sign of each term should be changed, not just the first sign. For example,

$$-\frac{a-b}{c-d}=+\frac{-(a-b)}{c-d}=\frac{-a+b}{c-d}=\frac{b-a}{c-d}.$$

c. **If the numerator or denominator is in factored form, change only the sign of one of the factors, not both. Thus,**

$$-\frac{(x-y)\ (x-2y)}{x+y}=\frac{(x+y)\ (x-2y)}{x+y}=\frac{(y-x)\ (x-2y)}{x+y}.$$

64. Changing Form of Algebraic Fractions

In algebra, as in arithmetic, any fraction can be changed to an equivalent fraction by multiplying or dividing both the numerator and denominator by the same term or number except zero. This will not change the value of the fraction. For example, to change the fraction $\frac{3}{5}$ to a fraction with 10 as its denominator, multiply both the numerator and the denominator by 2. Thus,

$$\frac{3}{5}=\frac{3\cdot 2}{5\cdot 2}=\frac{6}{10}.$$

Similarly, to change the fraction $\frac{x}{y}$ to a fraction with yz as its denominator, the denominator is changed to yz by multiplying by z; the numerator also is multiplied by z to become xz. Thus,

$$\frac{x}{y}=\frac{x\cdot z}{y\cdot z}=\frac{xz}{yz}.$$

Example 1: Change $\frac{4}{a-3}$ to a fraction with a^2-9 as its denominator.

$$\frac{4}{a-3}=\frac{4\cdot(a+3)}{(a-3)\ (a+3)}$$
$$=\frac{4(a+3)}{a^2-9}$$

Example 2: Change $\frac{4r-3}{6r}$ to a fraction with $18\pi r^2s$ as its denominator.

$$\frac{4r-3}{6r}=\frac{(4r-3)\cdot 3\pi rs}{6r\cdot 3\pi rs}=\frac{3\pi rs(4r-3)}{18\pi r^2s}$$

65. Reducing Fractions to Lowest Terms

As in arithmetic, when the numerator and denominator of a fraction have no common factor other than 1, the fraction is said to be in its lowest terms. The fraction $\frac{3}{8}$, $\frac{a}{b}$, and $\frac{p+q}{p-q}$, therefore, are in their lowest terms since the numerator and denominator of each fraction have no other factor except 1. The fractions $\frac{6}{12}$ and $\frac{3a}{9a^2}$ are not in their lowest terms. The fraction $\frac{6}{12}$ can be reduced to its lowest term by dividing both the numerator and denominator by 6. Similarly, the fraction $\frac{5y}{15y^2}$ can be reduced to $\frac{1}{3y}$ by dividing the numerator and denominator by $5y$. Thus, to reduce a fraction to its lowest terms, factor the numerator and denominator into prime factors and cancel the factors common to both (since they are equal to $\frac{1}{1}$).

Example 1: Reduce $\frac{6y}{8y^2}$ to lowest terms.

$$\frac{6y}{8y^2}=\frac{2y(3)}{2y(4y)}=\frac{3}{4y}$$

Example 2: Reduce $\frac{xab^2}{xcb}$ to lowest terms.

$$\frac{xab^2}{xcb}=\frac{xb\ (ab)}{xb\ (c)}=\frac{ab}{c}$$

Example 3: Reduce $\frac{a^2 - b^2}{4a + 4b}$ to lowest terms.

$$\frac{a^2 - b^2}{4a + 4b} = \frac{(a + b)\ (a - b)}{4(a + b)} = \frac{a - b}{4}$$

Example 4: Reduce $\frac{2a^2 + 4ab + 2b^2}{2a + 2b}$ to lowest terms.

$$\frac{2a^2 + 4ab + 2b^2}{2a + 2b} = \frac{2\ (a+b)\ (a+b)}{2(a + b)} = \frac{a + b}{1} = a + b$$

66. Finding Lowest Common Denominator

The lowest common denominator (LCD) of two or more fractions is the smallest term or number that is divisible by each of the denominators. Inspect to find this term or number, divide the LCD by the denominator of each fraction, and multiply both the numerator and denominator by the quotient. For example, when changing the fractions $\frac{2}{3}$ and $\frac{4}{5}$ to fractions which have an LCD, inspection shows that 15 is the smallest number which is divisible by both 3 and 5. Thus, the fractions $\frac{2}{3}$ and $\frac{3}{5}$ become $\frac{10}{15}$ and $\frac{9}{15}$. Similarly, the LCD of $\frac{4xy}{3a^2}$ and $\frac{6z}{4ab}$ is $12a^2b$ because this is the smallest term that is divisible by both $3a^2$ and $4ab$. Thus, the fraction $\frac{4xy}{3a^2}$ and $\frac{6z}{4ab}$ become $\frac{16xyb}{12a^2b}$ and $\frac{18za}{12a^2b}$, respectively. When fractions have factors with exponents in the denominators, the highest power of each distinct factor is used to form the LCD. For example, consider the problem of finding the LCD of fractions having the following denominators: x^3y^2z, $x^2y^3z^2$, y^4z^3, x^2y^4. The LCD is $x^3y^4z^3$ because x^3, y^4, and z^3 are the highest powers of x, y, and z in any one denominator.

Example: Change $\frac{3a}{a^2 - b^2}$ and $\frac{4b}{a^2 - ab - 2b^2}$ to equivalent fractions having an LCD.

Step 1. Factor each denominator into its prime factors:

$$\frac{3a}{a^2 - b^2} = \frac{3a}{(a + b)\ (a - b)}$$

$$\frac{4b}{a^2 - ab - 2b^2} = \frac{4b}{(a + b)\ (a - 2b)}$$

Step 2. The lowest common multiple of the denominators is the LCD:

$$(a + b)\ (a - b)\ (a - 2b)$$

Step 3. Divide the LCD by the denominators:

$$(a + b)\ (a - b)\ (a - 2b) \div (a + b)\ (a - b) = a - 2b$$

$$(a + b)\ (a - b)\ (a - 2b) \div (a + b)\ (a - 2b) = a - b$$

Step 4. Change $\frac{3a}{(a + b)\ (a - b)}$ into a fraction having $(a + b)$ $(a - b)(a - 2b)$ as its denominator:

$$\frac{3a}{(a + b)\ (a - b)} = \frac{3a\ (a - 2b)}{(a + b)\ (a - b)\ (a - 2b)}$$

Step 5. Change $\frac{4b}{(a+b)(a-2b)}$ into a fraction having $(a+b)(a-b)(a-2b)$ as its denominator.

$$\frac{4b}{(a+b)(a-2b)} = \frac{4b(a-b)}{(a+b)(a-b)(a-2b)}$$

Step 6. Therefore, $\frac{3a}{a^2-b^2} = \frac{3a(a-2b)}{(a+b)(a-b)(a-2b)}$

and $\frac{4b}{a^2-ab-b^2} = \frac{4b(a-b)}{(a+b)(a-b)(a-2b)}$

67. Addition and Subtraction of Algebraic Fractions

a. Addition. The addition of algebraic fractions is similar to the corresponding operation in arithmetic. To add two or more fractions having a common denominator, add the numerators and place the result over the common denominator. If the fractions have different denominators, convert them to fractions with an LCD. The sum of the fractions is equal to the algebraic sum of the numerators divided by the LCD. Simplify the numerator and reduce the result to its lowest terms. If possible, factor or combine for further simplification.

Example: Find the sum of $\frac{2x}{x+y}$ and $\frac{2y}{x-y}$.

The LCD is $(x+y)(x-y)$. Therefore,

$$\frac{2x}{x+y} + \frac{2y}{x-y} = \frac{2x(x-y)}{(x+y)(x-y)} + \frac{2y(x+y)}{(x+y)(x-y)}$$

$$= \frac{2x(x-y)+2y(x+y)}{(x+y)(x-y)}$$

$$= \frac{2x^2 - \cancel{2xy} + \cancel{2xy} + 2y^2}{(x+y)(x-y)}$$

$$= \frac{2x^2+2y^2}{(x+y)(x-y)}$$

$$= \frac{2(x^2+y^2)}{x^2-y^2}$$

b. Subtraction. To subtract two fractions having a common denominator, subtract the numerator of the subtrahend from the numerator of the minuend and place the result over the common denominator. If the denominators are different, find the LCD and subtract, as shown below.

Example: Subtract $\frac{8}{x^2+6x-16}$ from $\frac{9}{x^2+7x-18}$.

The LCD is $(x-2)(x+8)(x+9)$. Therefore,

$$\frac{9}{x^2+7x-18} - \frac{8}{x^2+6x-16}$$

$$= \frac{9(x+8)}{(x-2)(x+8)(x+9)} - \frac{8(x+9)}{(x-2)(x+8)(x+9)}$$

$$= \frac{9(x+8)-8(x+9)}{(x-2)(x+8)(x+9)}$$

$$= \frac{9x + \cancel{72} - 8x - \cancel{72}}{(x-2)(x+8)(x+9)}$$

$$= \frac{x}{(x-2)(x+8)(x+9)}$$

68. Multiplication and Division of Algebraic Fractions

a. Multiplication. The process of multiplication of algebraic fractions is the same as in arithmetic. The product of two or more fractions is the product of the numerators divided by the product of the denominators. The operation may be simplified by dividing common factors in the numerator and denominator by the same factor.

Example 1: Multiply $\frac{6a^2b}{7x}$ by $\frac{21x^2y}{24a^2b}$.

The first numerator and the second denominator are divisible by $6a^2b$; the first denominator and the second numerator are divisible by $7x$. Therefore:

$$\frac{\overset{1}{\cancel{6a^2b}}}{\underset{1}{\cancel{7x}}}\cdot\frac{\overset{3xy}{\cancel{21x^2y}}}{\underset{4}{\cancel{21a^2b}}}=\frac{3xy}{4}$$

Example 2: Multiply $\frac{a^2+2ab+b^2}{a-b}$ by $\frac{a^2-2ab+b^2}{a+b}$

$$\frac{a^2+2ab+b^2}{a-b}\cdot\frac{a^2-2ab+b^2}{a+b}=\frac{(a+b)(a+b)}{a-b}\cdot\frac{(a-b)(a-b)}{a+b}$$

$$=\frac{\overset{1}{\cancel{(a+b)}}(a+b)\overset{1}{\cancel{(a-b)}}(a-b)}{\underset{1}{\cancel{(a-b)}}\underset{1}{\cancel{(a+b)}}}$$

$$=(a+b)(a-b)$$

$$=a^2-b^2$$

b. Division. To divide algebraic fractions, multiply the dividend by the reciprocal of the divisor. Thus, to divide by x, multiply by the reciprocal of x, that is $\frac{1}{x}$. In other words, invert the divisor and proceed as in multiplication.

Example 1: Divide $\frac{2a+2b}{a-3}$ by $\frac{a^2-b^2}{2a-6}$.

$$\frac{2a+2b}{a-3}\div\frac{a^2-b^2}{2a-6}=\frac{2a+2b}{a-3}\cdot\frac{2a-6}{a^2-b^2}$$

$$=\frac{2\overset{1}{\cancel{(a+b)}}}{\underset{1}{\cancel{a-3}}}\cdot\frac{2\overset{1}{\cancel{(a-3)}}}{\underset{1}{\cancel{(a+b)}}(a-b)}$$

$$=\frac{2\cdot 2}{a-b}$$

$$=\frac{4}{a-b}$$

Example 2: Divide $\frac{z^2-z-6}{z^2-25}$ by $\frac{z^2+z-12}{z^2-z-20}$.

$$\frac{z^2-z-6}{z^2-25}\div\frac{z^2+z-12}{z^2-z-20}=\frac{z^2-z-6}{z^2-25}\cdot\frac{z^2-z-20}{z^2+z-12}$$

$$=\frac{\overset{1}{\cancel{(z-3)}}(z+2)}{\underset{1}{\cancel{(z-5)}}(z+5)}\cdot\frac{\overset{1}{\cancel{(z-5)}}\overset{1}{\cancel{(z+4)}}}{\underset{1}{\cancel{(z+4)}}\underset{1}{\cancel{(z-3)}}}$$

$$=\frac{z+2}{z+5}$$

69. Review Problems—Algebraic Fractions

a. *Changing Signs of Fractions.* Solve for the unknown.

(1) $\frac{4x+3}{6} - \frac{x-9}{4} = 5$

(2) $\frac{x-2}{4} = \frac{1}{2}$

(3) $\frac{r+4}{3} - \frac{r-2}{5} = 2$

(4) $\frac{4x-3}{6x} - \frac{4x+5}{3x} = 2$

(5) $\frac{7t+2}{3} = 3$

(6) $\frac{x-4}{3} + \frac{2x-5}{6} = 3$

(7) $\frac{2r+3}{2} - \frac{3r+2}{4} = 2$

(8) $\frac{7x-4}{3} + \frac{x-5}{5} = \frac{1}{5}$

b. *Equivalent Fractions.* Supply missing terms.

(1) $\frac{4}{8} = \frac{?}{16}$

(2) $\frac{1}{c} = \frac{?}{cx}$

(3) $\frac{3}{r-s} = \frac{?}{r^2-s^2}$

(4) $\frac{a-s}{1} = \frac{?}{3}$

(5) $\frac{I-6}{I-3} = \frac{?}{(I-3)(I-9)}$

(6) Change $\frac{4E^2}{R}$ into an equivalent fraction of which the denominator is $2I^2R$.

(7) Change $\frac{1}{3\pi fc}$ into an equivalent fraction of which the denominator is $2I^2R$.

c. *Lowest Common Denominator.* Reduce to equivalent fractions having an LCD.

(1) $\frac{1}{R}, \frac{1}{R^2}, \frac{1}{r}$

(2) $\frac{1}{a+1}, \frac{x}{a-1}$

(3) $\frac{b}{2x}, \frac{c}{3x}$

(4) $\frac{y}{2}, \frac{y}{2y+6}$

(5) $\frac{2}{c}, \frac{3}{c+1}$

(6) $\frac{i}{e-5}, \frac{i}{2e-10}$

(7) $\frac{y}{c^2-d^2}, \frac{z}{c-d}$

d. *Addition and Subtraction of Fractions.* Perform the indicated operations.

(1) $\frac{1}{a} + \frac{4}{a} + \frac{7}{a}$

(2) $\frac{s}{t} + \frac{s+4}{2t} + \frac{s+3}{4t}$

(3) $\frac{3a}{4x^2y} + \frac{5b}{6xy^3}$

(4) $\frac{2}{z^2-1} + \frac{4}{z^2-4}$

(5) $\frac{3c-2d}{4cd^2} + \frac{2c-3d}{3c^2d}$

(6) $\frac{(r+1)(r-3)}{r^2+2r-15} + \frac{(r-2)(r+5)}{r^2+2r-15}$

(7) $3y - \frac{1}{4}$

(8) $\frac{a+b}{a-b} - \frac{a-b}{a+b}$

(9) $\frac{32}{25q^2} - \frac{16}{5q}$

(10) $\frac{3t-2t}{4tv^2} - \frac{2t-3t}{3t^2v}$

e. *Multiplication and Division of Fractions.* Perform the indicated operations.

(1) $\frac{9y^2}{16} \cdot \frac{2}{3}$

(2) $\frac{a^3}{b^4} \cdot \frac{a^6}{b^2}$

(3) $\frac{3x^2}{49y^2z} \cdot \frac{7yz^2}{9xm}$

(4) $\left(\frac{1}{r} - \frac{1}{s}\right)\left(r - \frac{r^2}{s}\right)$

(5) $\frac{2x^2-5xy-3y^2}{x^2-9y^2} \cdot \frac{3x+9y}{10x^2+5xy}$

(6) $\frac{a-b}{a^2+2ab+b^2} \cdot \frac{a+b}{a^2-2ab+b^2} \cdot \frac{a^2-b^2}{a^2}$

(7) $3z \div \frac{1}{5}$

(8) $\frac{5ba^3}{6cd} \div 5b$

(9) $\frac{12s^2t}{20uv} \div \frac{3st}{4u^2v}$

(10) $\left(e+2-\frac{3}{e}\right) \div \left(e+1-\frac{2}{e}\right)$

70. General

Chapter 4 presents exponents and roots consisting only of whole numbers. However, to use exponents and radicals to solve many equations and formulas, a knowledge of additional operations is required.

71. Fractional Exponents

a. General. A fractional exponent is merely another way of expressing the root of a number. For example, the cube root of x usually is written $\sqrt[3]{x}$; however, it also can be written $x^{\frac{1}{3}}$. Similarly, $\sqrt{2}$ also can be written $2^{\frac{1}{2}}$.

b. Application. Fractional exponents have a practical value in simplifying algebraic problems. They follow the same rules as exponents that consist of integers, and can be added, subtracted, multiplied, or divided in the same way; thus

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a, \text{ and } a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a.$$

In other words, $a^{\frac{1}{2}}$ is one of two equal factors of a or the square root of a, and $a^{\frac{2}{3}}$ is two of three equal factors of a or the square cube root of a; therefore, $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{2}{3}} = \sqrt[3]{a^2}$.

c. Changing from Radical Form to Exponential Form. To change a radical expression to exponential form, remove the radical sign and annex a fractional exponent to the radicand (number under the radical sign). The numerator of the fractional exponent is the power of the radicand, and the denominator is the index of the root.

Example 1: Change $\sqrt[4]{a^2}$ to exponential form and simplify.

$$\sqrt[4]{a^2} = (a^2)^{\frac{1}{4}}$$

Multiplying exponents and simplifying:

$$(a^2)^{\frac{1}{4}} = a^{2 \cdot \frac{1}{4}} = a^{\frac{2}{4}} = a^{\frac{1}{2}} = \sqrt{a}$$

Therefore, $\sqrt[4]{a^2} = \sqrt{a}$

Example 2: Change $\sqrt[3]{8a^2b^3}$ to exponential form and simplify.

$$\sqrt[3]{8a^2b^3} = \sqrt[3]{2^3a^2b^3} = (2^3a^2b^3)^{\frac{1}{3}} = 2^{3 \cdot \frac{1}{3}} a^{2 \cdot \frac{1}{3}} b^{3 \cdot \frac{1}{3}}$$
$$= 2^{\frac{3}{3}}a^{\frac{2}{3}}b^{\frac{3}{3}} = 2^1a^{\frac{2}{3}}b^1 = 2b\,(a^2)^{\frac{1}{3}} = 2b\sqrt[3]{a^2}$$

d. Changing from Exponential Form to Radical Form. To change an expression with a fraction exponent to a radical form, make the base of the fractional exponent the radicand, the numerator of the exponent the power of the radicand, and the denominator of the exponent the index of the root.

Example 1: Change $4^{\frac{1}{2}}$ to radical form.

$$4^{\frac{1}{2}} = \sqrt{4}$$

Example 2: Change $3^{\frac{2}{3}}$ to radical form.

$$3^{\frac{2}{3}} = \sqrt[3]{3^2} = \sqrt[3]{9}$$

Example 3: Change $(5a^2b)^{\frac{2}{5}}$ to radical form.

$$(5a^2b)^{\frac{2}{5}} = \sqrt[5]{(5a^2b)^2}$$
$$= \sqrt[5]{25a^4b^2}$$

72. Simplification of Radicals

a. Removing a Factor from the Radicand. The form in which a radical expression is written may be changed without altering its numerical value. Sometimes there is a question as to what actually is the simplest form for an expression. For instance, consider the simplification of an expression such as $\sqrt{1250}$: $\sqrt{1250} = \sqrt{2.5^4} = 5^2\sqrt{2} = 25\sqrt{2}$. The expression $25\sqrt{2}$ usually is accepted as being simpler than $\sqrt{1250}$. As a general rule, the fewer the factors under the radical sign, the simpler the expression. Thus, a radicand may be separated into two factors, one of which is the greater power whose root can be taken. The root of this factor may then be written as the coefficient of a radical of which the other factor is the radicand.

Example 1: Simplify $\sqrt{50}$.

$$\sqrt{50} = \sqrt{25.2}$$
$$= \sqrt{25}.\sqrt{2}$$
$$= 5\sqrt{2}$$

Example 2: Simplify $\sqrt[4]{32a^7b^3}$.

$$\sqrt[4]{32a^7b^3} = (2^5a^7b^3)^{\frac{1}{4}}$$
$$= 2^{\frac{5}{4}}a^{\frac{7}{4}}b^{\frac{3}{4}}$$
$$= 2^{\frac{4}{4}}2^{\frac{1}{4}}a^{\frac{4}{4}}a^{\frac{3}{4}}b^{\frac{3}{4}}$$
$$= 2a\sqrt[4]{2a^3b^3}$$

b. Rationalizing Denominator. Rationalizing a denominator containing a radical means to eliminate the radical in the denominator. For example, to rationalize the expression $\frac{1}{\sqrt[3]{2}}$, first change the denominator into an expression having a fractional exponent; thus, $\frac{1}{\sqrt[3]{2}} = \frac{1}{2^{\frac{1}{3}}}$; then multiply the denominator by a number that will make its exponent equal to 1. This operation eliminates the radical sign below the line. In this case, $2^{\frac{2}{3}}$ is such a factor; thus $2^{\frac{1}{3}} \cdot 2^{\frac{2}{3}} = 2^1 = 2$. Such multiplication can be performed without changing the value of the fraction if the numerator also is multiplied by the same number; thus $\frac{1}{2^{\frac{1}{3}}} \cdot \frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}} = \frac{2^{\frac{2}{3}}}{2^{\frac{1}{3}+\frac{2}{3}}} = \frac{2^{\frac{2}{3}}}{2}$. Finally, changing the numerator into radical form, $\frac{\sqrt[3]{2^2}}{2} = \frac{\sqrt[3]{4}}{2}$. Therefore, to rationalize a denominator, multiply both the numerator and the denominator by a number that will make the exponent in the denominator equal to 1; then simplify the radicand in the numerator. The examples below illustrate the method of rationalizing a few different types of denominators.

Example 1: Rationalize $\frac{1}{3^{\frac{2}{7}}}$.

$$\frac{1}{3^{\frac{2}{7}}} = \frac{1}{3^{\frac{2}{7}}} \cdot \frac{3^{\frac{5}{7}}}{3^{\frac{5}{7}}} = \frac{3^{\frac{5}{7}}}{3} = \frac{\sqrt[7]{3^5}}{3}$$

Example 2: Rationalize $\frac{1}{\sqrt{8}}$.

First simplify $\sqrt{8}$.

$$\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2} = 2 \cdot 2^{\frac{1}{2}}$$
$$\frac{1}{\sqrt{8}} = \frac{1}{2 \cdot 2^{\frac{1}{2}}} = \frac{2^{\frac{1}{2}}}{2 \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}} = \frac{\sqrt{2}}{4}$$

Example 3: Rationalize $\frac{1}{\sqrt{7}}$.

Here the square root in the denominator is being multiplied by itself, making the number a perfect square.

$$\frac{1}{\sqrt{7}} = \frac{1}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{7}}{\sqrt{7}\sqrt{7}} = \frac{\sqrt{7}}{7}$$

c. Practical Application. The processes of the simplication of radicals and rationalization of denominators are useful when computing decimals. It is necessary to know, however, that $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, etc. For example, consider the problem of evaluating $\frac{1}{\sqrt{2}}$. One way of evaluating this problem is to divide 1 by 1.414. This evaluation is a long-division problem of some length, however. A much more simple way is to rationalize—thus $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, and dividing 1.414 by 2 gives the result, 0.707.

73. Addition and Subtraction of Radicals

As discussed in paragraph 27*b*, terms that are alike in all respects, except for their coefficients, are called *similar terms*. Similarly, radicals that have the same index and the same radicand and differ only in their coefficients are called *similar radicals*. For example, $-5\sqrt{3}$, $2\sqrt{3}$, and $\sqrt{3}$ are similar radicals. Similar radicals may be added or subtracted in the same way that similar terms are added and subtracted. However, if the radicands are not alike and cannot be reduced to a common radicand, they are dissimilar and addition and subtraction can only be indicated; thus to add or subtract radicals, reduce them to their simplest form, then combine similar radicals, and indicate the addition or subtraction of dissimilar radicals.

Example 1: **Perform the indicated operations.**

$$4\sqrt{6} - 5\sqrt{6} - \sqrt{6} + 10\sqrt{6} = 8\sqrt{6}$$

Example 2: **Add.**

$$\begin{aligned}\sqrt{48a} + \sqrt{\frac{a}{3}} + \sqrt{3a} &= 4\sqrt{3a} + \tfrac{1}{3}\sqrt{3a} + \sqrt{3a} \\ &= \frac{16}{3}\sqrt{3a}\end{aligned}$$

Example 3: **Perform the indicated operations.**

$$\begin{aligned}\sqrt[6]{16r^2} - r\sqrt[3]{4r} + \sqrt[9]{64r^3} &= \sqrt[6]{(4r)^2} - r\sqrt[3]{4r} + \sqrt[9]{(4r)^3} \\ &= (4r)^{\frac{2}{6}} - 4(4r)^{\frac{1}{3}} + (4r)^{\frac{3}{9}} \\ &= \sqrt[3]{4r} - r\sqrt[3]{4r} + \sqrt[3]{4r} \\ &= \sqrt[3]{4r}\,(2 - r)\end{aligned}$$

Example 4: **Perform the indicated operations.**

$$\begin{aligned}2\sqrt{6} + \sqrt[9]{\frac{2}{3}} - \sqrt[4]{36} &= 2\sqrt{6} + \sqrt[9]{\frac{2}{3}\ \frac{3}{3}} - \sqrt[4]{6.6} \\ &= 2\sqrt{6} + \frac{9}{3}\sqrt{6} - \sqrt[4]{6^2} \\ &= 2\sqrt{6} + 3\sqrt{6} - \sqrt{6} \\ &- 4\sqrt{6}\end{aligned}$$

74. Multiplication of Radicals

a. ***Radicals With Same Indexes.*** **Radicals can be multiplied and combined under the same radical sign even though they differ in value, provided the index of the radicals are the same. To multiply a radical expression when radicals are of the same order, first multiply the coefficients, then multiply the radicands, and then simplify, if possible. For example, $2\sqrt{3} \cdot 3\sqrt{5} = 6\sqrt{15}$. If the radicand is a perfect square, simplify the result by extracting the square root. Remember that there are two square roots, one positive and one negative; thus, $6\sqrt{3} \cdot 4\sqrt{3} = 24\sqrt{9} = 24(\pm 3) = \pm 72$. When polynomial expressions, either or both of which involve radicals, are to be multiplied, proceed in the same manner as with literal polynominal expressions (par. 48). For example, $(\sqrt{3} + 2\sqrt{5}) \times (\sqrt{3} - 2\sqrt{5}) =$**

$$\begin{array}{lll} \sqrt{3} + 2\sqrt{5} & & \\ \underline{\sqrt{3} - 2\sqrt{5}} & & \\ \sqrt{9} + 2\sqrt{15} & & \\ \underline{\quad - 2\sqrt{15} - 4\sqrt{25}} & & \\ \sqrt{9} \qquad\quad - 4\sqrt{25} & = & \pm 3 - 4(\pm 5) \\ & = & \pm 3 \pm 20 \\ & = & 3 \pm 20 \text{ or } -3 \pm 20 \\ & = & \pm 17 \text{ or } \pm 23 \end{array}$$

Example 1: **Multiply $2\sqrt[3]{3a}$, $5\sqrt[3]{4a}$, and $3\sqrt[3]{18a}$.**

$$\begin{aligned}2\sqrt[3]{3a} \cdot 5\sqrt[3]{4a} \cdot 3\sqrt[3]{18a} &= 2 \cdot 5 \cdot 3 \cdot \sqrt[3]{3a} \cdot \sqrt[3]{4a} \cdot \sqrt[3]{18a} \\ &= 30\sqrt[3]{216a^3} \\ &= 30 \cdot 6a \\ &= 180a\end{aligned}$$

Example 2: **Multiply $\sqrt[4]{8t^3}$ and $\sqrt[4]{4t^2s}$.**

$$\begin{aligned}\sqrt[4]{8t^3} \cdot \sqrt[4]{4t^2s} &= \sqrt[4]{32t^5s} \\ &= \sqrt[4]{2^4 \cdot 2 \cdot t^4 \cdot t \cdot s} \\ &= 2t\sqrt[4]{2ts}\end{aligned}$$

b. Radicals With Different Indexes. To multiply radicals when the indexes are different, first express them as radicals with a common index (or common fractional exponent) and proceed as in *a* above. The common index is the lowest common multiple of the indexes of the original radicals.

Example 1: Multiply $\sqrt{2} \cdot \sqrt[3]{4}$.

$$\sqrt{2} \cdot \sqrt[3]{4} = \sqrt{2} \cdot \sqrt[3]{2^2}$$
$$= 2^{\frac{1}{2}} \cdot 2^{\frac{2}{3}}$$
$$= 2^{\frac{3}{6}} \cdot 2^{\frac{4}{6}}$$
$$= 2^{\frac{7}{6}}$$
$$= 2^{\frac{6}{6}} \cdot 2^{\frac{1}{6}}$$
$$= 2 \cdot 2^{\frac{1}{6}} \text{ or } 2\sqrt[6]{2}$$

Example 2: Multiply $\sqrt[3]{4x} \cdot \sqrt[4]{8x^3}$.

$$\sqrt[3]{4x} \cdot \sqrt[4]{8x^3} = \sqrt[12]{(4x)^4} \cdot \sqrt[12]{(8x^3)^3}$$
$$= \sqrt[12]{(2^2x)^4 \cdot (2^3x^3)^3}$$
$$= \sqrt[12]{2^8 \cdot 2^9 \cdot x^4 \cdot x^9}$$
$$= \sqrt[12]{2^{17} \cdot x^{13}}$$
$$= \sqrt[12]{2^{12} \cdot 2^5 \cdot x^{12} \cdot x}$$
$$= 2x\sqrt[12]{2^5 \cdot x}$$
$$= 2x\sqrt[12]{32x}$$

75. Division of Radicals

a. Monomial Radical Expressions. The division of radicals is essentially the opposite of multiplication. When radicals are of the same order, the division of two radicals may be expressed under one radical sign—for example, $\frac{\sqrt{4}}{\sqrt{2}} = \sqrt{\frac{4}{2}} = \sqrt{2}$. When radicals are of different orders, they must be expressed as radicals having the same index or be changed to fractional exponents.

Example 1: Divide $\sqrt{15}$ by $\sqrt{5}$.

$$\frac{\sqrt{15}}{\sqrt{5}} = \sqrt{\frac{5}{15}} = \sqrt{3}$$

Example 2: Divide $\sqrt[3]{x^5y}$ by $\sqrt[3]{y^7}$.

$$\frac{\sqrt[3]{x^5y}}{\sqrt[3]{y^7}} = \sqrt[3]{\frac{x^5y}{y^7}}$$
$$= \sqrt[3]{\frac{x^5}{y^6}}$$
$$= \frac{x}{y^2}\sqrt[3]{x^2}$$

Example 3: Divide $\sqrt{35}$ by $\sqrt{15}$.

$$\frac{\sqrt{35}}{\sqrt{15}} = \sqrt{\frac{35}{15}}$$
$$= \sqrt{\frac{7}{3}}$$
$$= \frac{1}{3}\sqrt{21}$$

Example 4: Divide $\sqrt{4ab}\ \sqrt[3]{2ab}$ by $\sqrt[6]{4a^5b^3}$.

$$\frac{\sqrt{4ab}\ \sqrt[3]{2ab}}{\sqrt[6]{4a^5b^3}} = \frac{\sqrt[6]{(4ab)^3} \cdot \sqrt[6]{(2ab)^2}}{\sqrt[6]{4a^5b^3}}$$
$$= \sqrt[6]{\frac{64a^3b^3 \cdot 4a^2b^2}{4a^5b^3}}$$
$$= \sqrt[6]{64b^2}$$
$$= \sqrt[6]{2^6b^2} \text{ or } (2^6b^2)^{\frac{1}{6}}$$
$$= 2\sqrt[3]{b}$$

b. Binomial Expressions With Radical in Divisor. When the divisor is a binomial in which one or more of the terms contains a square root, division is performed by first rationalizing the divisor. Multiply the numerator and denominator of the fraction by the denominator with the sign between the terms changed; then simplify the numerator and the denominator.

Example 1: Divide 3 by $4 + \sqrt{6}$.

$$\frac{3}{4+\sqrt{6}} = \frac{3}{4+\sqrt{6}} \cdot \frac{4-\sqrt{6}}{4-\sqrt{6}}$$
$$= \frac{3(4-\sqrt{6})}{16-6}$$
$$= \frac{3}{10}\ (4-\sqrt{6})$$

Example 2: Divide $\sqrt{1+x} - \sqrt{1-x}$ by $\sqrt{1+x} + \sqrt{1-x}$.

$$\frac{\sqrt{1+x}-\sqrt{1=x}}{\sqrt{1+x}+\sqrt{1-x}} \quad \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \quad \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$= \frac{(1+x) - 2\sqrt{1-x^2} + (1-x)}{(1+x)-(1-x)}$$

$$= \frac{2-2\sqrt{1-x^2}}{2x}$$

$$= \frac{1-\sqrt{1-x^2}}{x}$$

76. Review Problems—Exponents and Radicals

a. **Simplify.**

(1) $2^{-\frac{1}{3}}(2^{\frac{2}{3}})$

(2) $(8^{\frac{2}{3}})^2$

(3) $\sqrt{50}$

(4) $\sqrt[3]{\frac{1}{16}}$

(5) $\sqrt{18x-9}$

(6) $\sqrt[n]{\frac{6x^{3n}}{y^n}}$

(7) $(x^{10}y^5)^{\frac{1}{5}}$

(8) $(d^6e^4)^{\frac{3}{4}}$

(9) $\left(\frac{64r^6}{s^3}\right)^{\frac{1}{3}}$

(10) $(a^9b^3)^{\frac{2}{3}}$

b. **Express with radical signs.**

(1) $4^{\frac{1}{3}}$

(2) $a^{\frac{3}{2}}b^{\frac{2}{3}}$

(3) $6^{\frac{2}{3}}$

(4) $(8f)^{\frac{1}{2}}$

(5) $5\,x^{.5}$

(6) $a^{\frac{3}{4}}c^{1.5}$

(7) $6r^{\frac{1}{3}}$

(8) $(8\,a^2b^3)^{\frac{1}{3}}$

(9) $({}^2r_1 + {}^3r_2)^{\frac{1}{2}}$

(10) $3(x^{\frac{1}{4}}y^2)^{\frac{1}{2}}$

c. **Express with fractional exponents.**

(1) $\sqrt[4]{a^3}$

(2) $\sqrt[3]{5x}$

(3) $6x\,\sqrt[3]{d^2}$

(4) $\sqrt[5]{z^2}$

(5) $\sqrt[4]{3a^3b^5}$

(6) $y^2\,\sqrt[4]{a^3}$

(7) $8\,\sqrt[3]{3e}$

(8) $9\sqrt[5]{g^4}$

(9) $3b\sqrt[6]{cd^2}$

(10) $\sqrt[3]{(x-y)^2}$

d. **Simplify by removing suitable factors from radicand.**

(1) $\sqrt{12}$

(2) $\sqrt{63}$

(3) $\sqrt{63x^2}$

(4) $2\sqrt{72a^2b^4}$

(5) $\sqrt{60b^2d^2}$

(6) $\sqrt{8I^2R}$

(7) $3\,\sqrt{63p^3z^2}$

(8) $2dr^2\,\sqrt{108dr^4s^3}$

(9) $5a\,\sqrt{81a^2b}$

(10) $16w^2x\,\sqrt{98w^4x^3y^2z}$

e. **Rationalize denominators.**

(1) $\frac{1}{\sqrt{50}}$

(2) $\frac{1}{\sqrt{4x}}$

(3) $\frac{2a}{\sqrt{3a}}$

(4) $\frac{1}{\sqrt[3]{x}}$

(5) $\frac{1}{\sqrt[4]{3ax^2}}$

(6) $\dfrac{1}{\sqrt[3]{3-2x}}$

(7) $\dfrac{a+b}{\sqrt[3]{a^2}}$

(8) $\dfrac{a}{\sqrt[3]{a^2bc}}$

(9) $\dfrac{1}{\sqrt[3]{(s+1)^2}}$

(10) $\dfrac{i+3}{\sqrt[5]{(i+3)^2}}$

f. Simplify.

(1) $6\sqrt{4} - 3\sqrt{4} + 2\sqrt{4}$

(2) $6\sqrt{45} - 2\sqrt{20}$

(3) $x - \sqrt{\dfrac{3x^2}{4}}$

(4) $\dfrac{a}{2} + \sqrt{\dfrac{9a^2}{2}}$

(5) $r\sqrt{rst} + rt\sqrt{\dfrac{s}{rt}}$

(6) $\sqrt{\dfrac{x+y}{x-y}} - \sqrt{\dfrac{x-y}{x+y}}$

(7) $\sqrt{5} + 3\sqrt{x} + 5\sqrt{x}$

(8) $7\sqrt{a} - 4\sqrt{b} - 2\sqrt{b}$

(9) $4\sqrt{x-y} + 3\sqrt{x+y} - 8\sqrt{x-y}$

(10) $3\sqrt{125a^3b^2} + b\sqrt{20a^3} - \sqrt{500a^3b^2}$

g. Find product and simplify.

(1) $3\sqrt{5} \cdot 4\sqrt{2}$

(2) $2\sqrt[3]{9} \cdot 3\sqrt[3]{3}$

(3) $4\sqrt[3]{a^2b^4} \cdot 2\sqrt[3]{ab^2}$

(4) $\sqrt{4z^2} \cdot z\sqrt{3z^3}$

(5) $\sqrt[5]{4x^2y^2} \cdot \sqrt[5]{2x^3y^2} \cdot \sqrt[5]{4xy^3}$

(6) $2\sqrt[3]{2pq^2r} \cdot \sqrt[3]{4pq^3r^2} \cdot 3\sqrt[3]{8pq^2r^3}$

(7) $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$

(8) $a\sqrt{x}(a\sqrt{ax} + x\sqrt{ax} + \sqrt{ax})$

(9) $\sqrt{9 - \sqrt{17}} \cdot \sqrt{9 + \sqrt{17}}$

(10) $\sqrt[3]{x^3y^6\sqrt{256a^8}}$

h. Divide and simplify.

(1) $\dfrac{\sqrt{12}}{\sqrt{8}}$

(2) $\dfrac{\sqrt[3]{625y}}{\sqrt[3]{5y}}$

(3) $\dfrac{\sqrt[3]{16x^2}}{\sqrt[3]{2x}}$

(4) $\dfrac{3zy}{\sqrt{zv}}$

(5) $\dfrac{2}{\sqrt{6-2}}$

(6) $\dfrac{\sqrt{30a}\ \sqrt[4]{24a^2}\ \sqrt[3]{72a}}{\sqrt[6]{5a}}$

(7) $\dfrac{\sqrt{2} + \sqrt{c}}{\sqrt{c} + 2\sqrt{2}}$

(8) $\dfrac{4\sqrt{3} - 3\sqrt{2}}{\sqrt{6}} \div \dfrac{\sqrt{10}}{4\sqrt{3} + 3\sqrt{2}}$

(9) $\dfrac{\sqrt{e^2 + f^2} + f}{\sqrt{e^2 + f^2} - f}$

(10) $\dfrac{2b + \sqrt{1 - 4b^2}}{2b - \sqrt{1 - 4b^2}}$

Section VII. IMAGINARY AND COMPLEX NUMBERS

77. Imaginary Numbers

a. Indicated Square Root of Negative Numbers.

(1) In the study of roots to this point, only the roots of positive numbers have been considered. Sometimes a negative expression will appear under the radical. Such an expression originally was given the designation *imaginary number* to distinguish it from real numbers. In electricity and electronics, however, so-called imaginary numbers are used for real physical calculations—the reactance of a large capacitor or inductor must be calculated by using this type of number.

(2) In multiplication, when a real number is multiplied by itself the result is always positive. For example, $+5 \cdot +5 = 25$, and $-5 \cdot -5 = 25$. Therefore, any number raised to a power having an even exponent will be positive because like signs are being multiplied. However, this is not true for the interpretation of an expression such as $\sqrt{-9}$. Any negative number can be regarded as the product of a positive number of the same absolute value and -1, and the square root of a negative

number can be written as the square root of a positive number times $\sqrt{-1}$; thus, $\sqrt{-9} = \sqrt{9}\ \sqrt{-1} = 3\sqrt{-1}$, with $\sqrt{-1}$ being the imaginary number. Most mathematics texts represent the imaginary number $\sqrt{-1}$ by the letter *i*. However, the letter *I* or *i* means current in electrical formulas; therefore, the letter *j*, commonly called the *operator j*, is used in electronics.

Example 1: $\sqrt{-36} = \sqrt{(-1)36} = \sqrt{-1} \cdot \sqrt{36} = \sqrt{-1} \cdot 6 = j6$

Example 2: $\sqrt{-Z^2} = \sqrt{(-1)Z^2} = \sqrt{-1} \cdot \sqrt{Z^2} = \sqrt{-1} \cdot Z = jZ$

Example 3: $-\sqrt{-9a^2} = -\sqrt{(-1)9a^2} = -\sqrt{-1} \cdot \sqrt{9a^2} = -\sqrt{-1} \cdot 3a = -j3a$

b. Powers of Operator j. Imaginary numbers follow the fundamental laws of addition, subtraction, multiplication, and division. They also can be raised to a power; thus, $j^3 = j^2 \cdot j = -1(j) = -j$, and $j^4 = j^2 \cdot j^2 = -1(-1) = 1$. The values of the powers of *j* are obtained as follows:

$j^2 = j \cdot j = \sqrt{-1} \cdot \sqrt{-1} = -1$;

$j^3 = j \cdot j \cdot j = \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} = -1\sqrt{-1} = -j$; and

$j^4 = j \cdot j \cdot j \cdot j = \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} \cdot \sqrt{-1} = -1 \cdot -1 = 1$; but

$j^5 = j \cdot j \cdot j \cdot j \cdot j = j^4 \cdot j = j^1 = \sqrt{-1}$, and the whole cycle starts over again. Therefore, j^4 can be eliminated as many times as it is contained in an expression, reducing the quantity to j, j^2, or j^3 and getting its value from the following:

$$j = j = \sqrt{-1}$$
$$j^2 = -1$$
$$j^3 = -j$$
$$j^4 = 1$$

Example 1: Simplify j^{13}.

$$j^{13} = j^{12} \cdot j = j = \sqrt{-1}$$

Example 2: Simplify j^{27}.

$$j^{27} = j^{24} \cdot j^3 = j^3 = -j = -\sqrt{-1}$$

c. Addition and Subtraction of Imaginary Numbers. These numbers may be added or subtracted in the same manner that any algebraic expression is added or subtracted (par. 44). First change the expression to the *j* form; then treat the *j* as any other letter in an algebraic expression.

Example 1: Add $\sqrt{-25}$, $\sqrt{-36}$, and $\sqrt{-9}$.

$$\sqrt{-25} + \sqrt{-36} + \sqrt{-9} = j5 + j6 + j3 = j14$$

Example 2: Add $6\sqrt{-2} + 5\sqrt{-8} + 8\sqrt{-18}$.

$$\begin{aligned} 6\sqrt{-2} + 5\sqrt{-8} + 8\sqrt{-18} &= j^6\sqrt{2} + j^5\sqrt{8} + j^8\sqrt{18} \\ &= j^6\sqrt{2} + j(5 \cdot 2)\sqrt{2} + j(8 \cdot 3)\sqrt{2} \\ &= (j^6 + j^{10} + j^{24})\sqrt{2} \\ &= j^{40}\sqrt{2} \end{aligned}$$

Example 3: Subtract $\sqrt{-64}$ from $\sqrt{-36}$.

$$\sqrt{-36} - \sqrt{-64} = j^6 - j^8 = -j^2$$

Example 4: Subtract $4\sqrt{-8}$ from $6\sqrt{-18}$.

$$\begin{aligned} 6\sqrt{-18} - 4\sqrt{-8} &= j(6 \cdot 3)\sqrt{2} - j(4 \cdot 2)\sqrt{2} \\ &= (j^{18} - j^8)\sqrt{2} \\ &= j^{10}\sqrt{2} \end{aligned}$$

d. Multiplication of Simple Imaginary Numbers. When multiplying two imaginary numbers, remember that $j^2 = -1$, $j^3 = -j$, and $j^4 = 1$ (*b* above); then, proceed as with any problem in multiplication (par. 45).

Example 1: Multiply $\sqrt{-16}$ and $\sqrt{-4}$.

$$\sqrt{-16} \cdot \sqrt{-4} = j^4 \cdot j^2 = j^2 8 = (-1)8 = -8$$

Example 2: Multiply $\sqrt{-81}$, $\sqrt{-25}$, and $\sqrt{-49}$.

$$\sqrt{-81} \cdot \sqrt{-25} \cdot \sqrt{-49} = j^9 \cdot j^5 \cdot j^7 = j^3 315 = (-j)315 = -j315$$

e. Division of Single Imaginary Numbers. In the division of two simple imaginary numbers, when both the dividend and divisor contain operator j, divide both by j and proceed as with ordinary integers. If a j remains in the denominator, the denominator must be rationalized because the j represents a radical expression. To rationalize, multiply both the numerator and denominator by the imaginary number.

Example 1: Divide $\sqrt{-100}$ by $\sqrt{-16}$.

$$\frac{\sqrt{-100}}{\sqrt{-16}} = \frac{\overset{1}{\cancel{j}} \cdot 10}{\underset{1}{\cancel{j}} \cdot 4} = 2\tfrac{1}{2}$$

Example 2: Divide 12 by $\sqrt{-6}$.

$$\frac{12}{\sqrt{-6}} = \frac{12}{j\sqrt{6}} = \frac{12 \cdot j\sqrt{6}}{j\sqrt{6} \cdot j\sqrt{6}} = \frac{j12\sqrt{6}}{j^2 6} = \frac{j2\sqrt{6}}{-1} = -j2\sqrt{6}$$

Example 3: Divide $\sqrt{-3}$ by $\sqrt{-4}$.

$$\frac{\sqrt{-3}}{\sqrt{-4}} = \frac{\overset{1}{\cancel{j}}\sqrt{3}}{\underset{1}{\cancel{j}}2} = \frac{\sqrt{3}}{2} \text{ or } \frac{1}{2}\sqrt{3}$$

Example 4: Divide 6 by j.

$$\frac{6}{j} = \frac{6}{j} \cdot \frac{j}{j} = \frac{j6}{j2} = \frac{j6}{-1} = -j6$$

78. Complex Numbers

a. Operations With Complex Numbers. A *complex number* is a real number united to an imaginary number by a plus or minus sign; thus, $10 - j5$, $x + jy$, and $R + jx$ are complex numbers. Complex numbers are of great importance in alternating-current electricity in which many problems would be difficult to solve without their use. A complex number expressed in the form $x + jy$ may be considered a binomial; thus, the addition, subtraction, multiplication, and division of complex numbers are reduced to the corresponding operations with binomials in which one term is real and the other imaginary.

b. Addition and Subtraction of Complex Numbers. To add or subtract complex numbers, first combine the real parts, then combine the imaginary parts, and write the results as a binomial with the appropriate sign separating the real and imaginary terms.

Example 1: Add $3 + j5$ and $5 - j$.

$$(3 + j5) + (5 - j) = 3 + j5 + 5 - j$$
$$= 8 + j4$$

Example 2: Add $6 + \sqrt{-25}$ and $8\sqrt{-16}$.

$$
\begin{aligned}
(6 + \sqrt{-25}) + (8\sqrt{-16}) &= 6 + j5 + (8 \cdot j4) \\
&= 6 + j5 + j32 \\
&= 6 + j37
\end{aligned}
$$

Example 3: Add $8 + \sqrt{-12}$ and $9 + \sqrt{-75}$.

$$
\begin{aligned}
(8 + \sqrt{-12}) + (9 + \sqrt{-75}) &= 8 + j2\sqrt{3} + 9 + j5\sqrt{3} \\
&= 17 + j7\sqrt{3}
\end{aligned}
$$

Example 4: Subtract $7 - j6$ from $3 - j2$.

$$
\begin{aligned}
(3 - j2) - (7 - j6) &= 3 - j2 - 7 + j6 \\
&= -4 + j4
\end{aligned}
$$

Example 5: Subtract $2 - 3\sqrt{-4}$ from $10 + \sqrt{-4}$.

$$
\begin{aligned}
(10 + \sqrt{-4}) - (2 - 3\sqrt{-4}) &= (10 + j2) - (2 - j6) \\
&= 10 + j2 - 2 + j6 \\
&= 8 + j8 \text{ or } 8(1 + j)
\end{aligned}
$$

Example 6: Subtract $3 + 7\sqrt{-24}$ from $5 + 3\sqrt{-6}$.

$$
\begin{aligned}
(5 + 3\sqrt{-6}) - (3 + 7\sqrt{-24}) &= 5 + j3\sqrt{6} - [3 + j(7 \cdot 2)\sqrt{6}] \\
&= 5 + j3\sqrt{6} - 3 - j14\sqrt{6} \\
&= 2 - j11\sqrt{6}
\end{aligned}
$$

c. Multiplication of Complex Numbers. As in addition and subtraction, when complex numbers are multiplied they are treated as ordinary binomials. Remember, however, that $j^2 = -1$.

Example 1: Multiply $3 - j6$ by $4 + j2$.

$$
\begin{array}{l}
3 - j6 \\
\underline{4 + j2} \\
12 - j24 \\
\underline{\quad + j6 - j^2 12} \\
12 - j18 - j^2 12
\end{array}
\begin{aligned}
&= j12 - j18 - (-1)(12) \\
&= 12 - j18 + 12 \\
&= 24 - j18
\end{aligned}
$$

Example 2: Multiply $8 - \sqrt{-5}$ by $-2 + \sqrt{-6}$.

$$
\begin{array}{l}
8 - j\sqrt{5} \\
\underline{-2 + j\sqrt{6}}
\end{array}
$$

$$
\begin{aligned}
-16 + j2\sqrt{5} + j8\sqrt{6} - j^2\sqrt{30} &= -16 + j2\sqrt{5} + j8\sqrt{6} - (-1)\sqrt{30} \\
&= -16 + j2\sqrt{5} + j8\sqrt{6} + \sqrt{30} \\
&= -16 + \sqrt{30} + j(2\sqrt{5} + 8\sqrt{6})
\end{aligned}
$$

d. Division of Complex Numbers. When dividing complex numbers, the denominator of the expression in its fractional form must first be rationalized (par. 74). To obtain a real number as a divisor, multiply both the numerator and denominator by the complex number of the denominator with its sign changed (called the *conjugate* of the complex number). In carrying out the multiplication, the radical expression is eliminated. Since $j^2 = -1$, the sign of the coefficient of j^2 is changed; the complex number thus becomes a real number to combine with the other real number in the denominator.

Example 1: Divide $3 + j4$ by $1 + j$.

$$\frac{3+j4}{1+j} = \frac{3+j4}{1+j} \cdot \frac{1-j}{1-j}$$

$$= \frac{3+j-j^2 4}{1-j^2}$$

$$= \frac{3+j-(1-1)4}{1-(-1)}$$

$$= \frac{3+j+4}{2}$$

$$= \frac{7}{2} + j\frac{1}{2}$$

Example 2: Divide 6 by $3 + \sqrt{-2}$.

$$\frac{6}{3+\sqrt{-2}} = \frac{6}{3+j\sqrt{2}} \cdot \frac{3-j\sqrt{2}}{3-j\sqrt{2}}$$

$$= \frac{6(3-j\sqrt{2})}{(3+j\sqrt{2})(3-j\sqrt{2})}$$

$$= \frac{18-j6\sqrt{2}}{9-j^2 2}$$

$$= \frac{18-j6\sqrt{2}}{11}$$

79. Review Problems—Imaginary and Complex Numbers

a. Simplify the radical, using operator j.

(1) $\sqrt{-75}$

(2) $\sqrt{-23}$

(3) $-\sqrt{-64ax^6}$

(4) $-\sqrt{-100x^6y^4}$

(5) $\sqrt{-\frac{1}{9}}$

(6) $\sqrt[3]{-128x^6y^6}$

b. Add.

(1) $-47 + j17$ and $63 + j92$

(2) $27 - j11$ and $14 - j11$

(3) $123 - j114$ and $-62 - j137$

(4) $44 + j17$ and $-j7$

(5) $6 + j10$ and $j1$

(6) $14 + j15$ and $-16 - j62$

c. Subtract.

(1) $-69 + j432$ from $710 + j61$

(2) $14 - j121$ from $73 - j7$

(3) $84 - j62$ from $62 - j47$

(4) $-74 - j20$ from $81 - j81$

(5) $-87 - j7$ from $82 + j16$

(6) $-9 + j$ from $-j7$

d. Multiply.

(1) $4 + \sqrt{-81}$ by $2 + \sqrt{-49}$

(2) $2 + 2\sqrt{-2}$ by $3 + 3\sqrt{-3}$

(3) $2 - j3$ by $2 + j3$

(4) $(2 - j3)^2$

(5) $(j^4 + j^3 2 + j^2 3 + j4)^2$

(6) $4 - j7$ by $8 + j2$

(7) $f + jg$ by $f + jg$

(8) $I + jE$ by $I - jE$

(9) $8 - j13$ by $11 - j12$

(10) $5 + \sqrt{-16}$ by $7 - \sqrt{-81}$

e. Divide.

(1) 1 by $3 + j2$

(2) $6 + j$ by j

(3) $2 + j3$ by $3 - j4$

(4) $4 + \sqrt{-9}$ by $2 - \sqrt{-1}$

(5) $x + jy$ by $x - jy$

(6) 10 by $1 + j2$

(7) 3 by $1 - j$

(8) $3 + \sqrt{-25}$ by $4 - \sqrt{-4}$

(9) $6 - j2$ by $4 - j7$

(10) $I + jE$ by $I - jE$

80. General

An *equation* is a statement of equality between two expressions. For example, $x + y = 12$, $3x + 5 = 20$, and $3 \cdot 9 = 27$ are equations; therefore, all expressions separated by the equality sign are equations, whether the expressions are algebraic or arithmetical. The expression to the left of the equality sign is called the *left-hand member* of the equation; the expression to the right of the equality sign is called the *right-hand member*. Finding the values of the unknown quantities of an algebraic equation is known as solving the equation, and the answer is called the *solution*. If only one unknown is involved, the solution is also called the *root*.

81. Solving Simple Equations

a. Adding Same Quantity to Both Members of Equation. Equal quantities may be added to both sides of an equation without changing the equality.

Example 1: Solve the equation $x - 4 = 7$ for x.

$$x - 4 = 7$$
$$x - 4 + 4 = 7 + 4$$
$$x = 11$$

Example 2: Solve the equation $x - 7 = 14$ for x.

$$x - 7 = 14$$
$$x - 7 + 7 = 14 + 7$$
$$x = 21$$

b. Subtracting Same Quantity From Both Members of Equation. Equal quantities may be subtracted from both sides of an equation.

Example 1: Solve the equation $x + 2 = 5$ for x.

$$x + 2 = 5$$
$$x + 2 - 2 = 5 - 2$$
$$x = 3$$

Example 2: Solve the equation $x + 5 = 12$ for x.

$$x + 5 = 12$$
$$x + 5 - 5 = 12 - 5$$
$$x = 7$$

c. Multiplying Both Members of Equation by Same Quantity. Both sides of an equation can be multiplied by the same quantity.

Example 1: Solve the equation $\frac{x}{3} = 5$ for x.

$$\frac{x}{3} = 5$$
$$\frac{x}{\cancel{3}} \cdot \frac{\cancel{3}}{1} = 5 \cdot 3$$
$$x = 15$$

Example 2: Solve the equation $\frac{z}{3} + \frac{z}{9} =$ 4 for z.

Multiply both sides of the equation by 9.

$$\left(\frac{z}{\cancel{3}} \cdot \frac{\overset{3}{\cancel{9}}}{1}\right) + \left(\frac{z}{\cancel{9}} \cdot \frac{\cancel{9}}{1}\right) = 4 \cdot 9$$
$$3z + z = 36$$
$$4z = 36$$
$$z = 9$$

d. Dividing Both Members of Equation by Same Quantity. Both sides of an equation may be divided by the same quantity.

Example 1: Solve the equation $3x = 12$ for x.

$$3x = 12$$
$$\frac{3x}{3} = \frac{12}{3}$$
$$x = 4$$

Example 2: Solve the equation $PV = RT$ for T.

$$PV = RT$$
$$\frac{PV}{R} = \frac{\cancel{R}T}{\cancel{R}}$$
$$T = \frac{PV}{R}.$$

82. Solving More Difficult Equations

a. Transposition. The process of adding to or subtracting from both members of an equation (par. 81*a* and *b*) can be shortened by shifting a term or terms from one side of the equation to the other and changing the signs. This operation is called transposition.

Example 1: Solve the equation $6x + 4 = x -$ 16 for x.

$$6x + 4 = x - 16$$
$$6x - x = -16 - 4$$
$$5x = -20$$
$$x = -4$$

Example 2: Solve the equation $5a - 7 = 2a + 2$ for a.

$$5a - 7 = 2a + 2$$
$$5a - 2a = 2 + 7$$
$$3a = 9$$
$$a = 3$$

b. Equations With Fractions. In solving a fractional equation, first find the LCD and multiply both members of the equation, term by term; then perform the operations in paragraph 81 or *a* above.

Example 1: Solve the equation $\frac{x}{2} + \frac{x}{3} = 10$ for x.

$$\frac{x}{2} + \frac{x}{3} = 10$$
$$\frac{3x + 2x}{6} = 10$$
$$\frac{5x}{6} = \frac{10}{1}$$
$$5x = 60$$
$$x = 12$$

Example 2: Solve the equation $\frac{x - 1}{2} = 3 + x$ for x.

$$\frac{x - 1}{2} = 3 + x$$
$$\frac{x - 1}{2} = \frac{3 + x}{1}$$
$$1(x - 1) = 2(3 + x)$$
$$x - 1 = 6 + 2x$$
$$x - 2x = 6 + 1$$
$$-x = 7$$
$$x = -7$$

Example 3: Solve the equation $\frac{2}{x - 2} + \frac{2}{x + 4} = \frac{4}{x - 3}$ for x.

$$\frac{2}{x - 2} + \frac{2}{x + 4} = \frac{4}{x - 3}$$
$$\frac{2(x + 4) + 2(x - 2)}{(x - 2)(x + 4)} = \frac{4}{x - 3}$$
$$\frac{2x + 8 + 2x - 4}{(x - 2)(x + 4)} = \frac{4}{x - 3}$$
$$\frac{4x + 4}{(x - 2)(x + 4)} = \frac{4}{x - 3}$$
$$(4x + 4)(x - 3) = 4(x - 2)(x + 4)$$
$$4x^2 - 8x - 12 = 4(x^2 + 2x - 8)$$
$$4x^2 - 8x - 12 = 4x^2 + 8x - 32$$
$$\cancel{4x^2} - \cancel{4x^2} - 8x - 8x = -32 + 12$$
$$-16x = -20$$
$$16x = 20$$
$$x = \frac{20}{16} = \frac{5}{4} = 1\frac{1}{4}$$
$$x = 1\frac{1}{4}$$

83. Written Equations

Many practical problems are stated in words and must be translated into symbols before the rules of algebra can be applied. There are no specific rules for the translation of a written problem into an equation of numbers, signs, and symbols. The following general suggestions may be helpful in developing equations:

a. From the worded statement of the problem, select the unknown quantity (or one of the unknown quantities) and represent it by a letter, such as x. Write the expression, stating exactly what x represents and the units in which it is measured.

b. If there is more than one unknown quantity in the problem, try to represent each unknown in terms of the first unknown.

Example 1: In simple problems, an equation may be written by an almost direct translation into algebraic symbols; thus,

$$\underbrace{\text{Seven}}_{7}\ \underbrace{\text{times}}_{\times}\ \underbrace{\text{a certain voltage}}_{E}\ \underbrace{\text{diminished by}}_{-}\ \underbrace{3}_{3}$$

$$\underbrace{\text{gives the same result as}}_{=}\ \underbrace{\text{the voltage}}_{}\ \underbrace{\text{increased by}}_{+}\ \underbrace{75,}_{75.}$$

Solving the equation:

$$7E - 3 = E + 75$$
$$7E - E = 75 + 3$$
$$6E = 78$$
$$E = 13$$

Check: $7(13) - 3 = 13 + 75$
$91 - 3 = 13 + 75$
$88 = 88$

Example 2: A triangle has a perimeter of 30 inches. The longest side is 7 inches longer than the shortest side, and the third side is 5 inches longer than the shortest side. Find the length of the three sides.

Let x = length of shortest side.
$x + 7$ = length of longest side.
$x + 5$ = length of third side.
$x + (x + 5) + (x + 7) = 30$
Solving the equation:
$x + x + 5 + x + 7 = 30$
$3x + 12 = 30$
$3x = 30 - 12$
$3x = 18$
$x = 6$ = shortest side.
$6 + 5 = 11$ = third side.
$6 + 7 = 13$ = longest side.

84. Simultaneous Equations

a. Definition. Simultaneous equations are two or more equations satisfied by the same sets of values of the unknown quantities. They are used to solve a problem with two or more unknown quantities.

b. Example. Assume that the sum of two numbers is 17, and that three times the first number less two times the second number is equal to 6. What are the numbers? In setting up equations for this problem, let x equal the first number and y equal the second number. The first equation is $x + y = 17$, and the second equation is $3x - 2y = 6$. This problem can be solved in three ways: by substitution, by addition, or by subtraction. All three methods are explained below.

(1) *Substitution.*
$x + y = 17$ or $x = 17 - y$
Substitute $x = 17 - y$ in the second equation:
$3x - 2y = 6$
$3(17 - y) - 2y = 6$
Remove the parentheses:
$51 - 3y - 2y = 6$
Transpose:
$-5y = 6 - 51$
$-5y = -45$
$5y = 45$
$y = 9$
Substitute $y = 9$ in the first equation and solve for x:
$x + y = 17$ or $x + 9 = 17$
Transpose:
$x = 17 - 9$
$x = 8$

(2) *Addition.*
$x + y = 17$
$3x - 2y = 6$
Before adding, change the y in the first equation to $2y$ so that the y terms drop out when added; thus, the first equation must be multiplied by 2.

$$\begin{array}{r} 2x + 2y = 34 \\ \underline{3x - 2y = \ \ 6} \\ 5x \qquad = 40 \\ x = \ \ 8 \end{array}$$

Substitute $x = 8$ in the first equation and solve for y:
$x + y = 17$ or $8 + y = 17$
$y = 17 - 8$
$y = 9$

(3) *Subtraction.*
Before subtracting, multiply the first equation by 3 so that the x terms drop out when subtracted.

$$\begin{array}{r} 3x + 3y = 51 \\ \underline{3x - 2y = \ \ 6} \end{array}$$

Subtract the second equation from the first equation:

$$3x + 3y = 51$$
$$\underline{-3x + 2y = -6}$$
$$5y = 45$$
$$y = 9$$

Substitute $y = 9$ in the first equation and solve for x: Refer to (1) and (2) above.

c. Additional Examples. If the coefficients of the unknowns differ (for example, $3x$ and x and $2y$ and $4y$), multiply one or both equations to establish equal coefficients for one of the unknowns (x or y).

Example 1: Solve for x and y if $3x + 2y = 7$ and $x + 4y = 9$.

$$3x + 2y = 7$$
$$\underline{x + 4y = 9}$$

Multiply the first equation by 2 so that $2y$ will become $4y$:

$$6x + 4y = 14$$
$$\underline{x + 4y = 9}$$

Subtract the second equation from the first equation:

$$6x + 4y = 14$$
$$\underline{-x - 4y = -9}$$
$$5x = 5$$
$$x = 1$$

Solve for y by substituting $x = 1$ in either equation.

Example 2: Solve for x and y if $2x + 3y = 24$ and $3x - 4y = 2$.

$$2x + 3y = 24$$
$$\underline{3x - 4y = 2}$$

Multiply the first equation by 4 to change $3y$ to $12y$; multiply the second equation by 3 to change $4y$ to $12y$; then add the two equations:

$$8x + 12y = 96$$
$$\underline{9x - 12y = 6}$$
$$17x = 102$$
$$x = 6$$

Solve for y by substituting $x = 6$ in either equation.

85. Solving Formulas

a. The Formula. A formula is a rule or law that states a scientific relationship. It can be expressed in an equation by using letters, symbols, and constant terms. For example, a formula in electricity (par. 184) states that the voltage across any part of a circuit is equal to the product of the current and resistance of that part of the circuit. In formula form, this is expressed as $E = IR$, where E is the *voltage or difference in potential* expressed in *volts*, I is the *current* expressed in *amperes*, and R is the *resistance* expressed in *ohms*.

b. Solving the Formula. To solve a formula, perform the same operations on both members of an equation until the desired unknown can be isolated in one member of the equation. If the numerical values for some variables are given, substitute in the formula and solve for the unknown as in any other equation.

Example 1: Solve the formula $T = \frac{12(D - d)}{l}$ for D.

$$T = \frac{12(D - d)}{l}$$
$$T = \frac{12D - 12d}{l}$$

Multiply both sides by l:

$$Tl = 12D - 12d$$

Transpose and change signs:

$$12D = Tl + 12d$$

Divide both sides by 12:

$$\frac{12D}{12} = \frac{Tl}{12} + \frac{12d}{12}$$
$$D = \frac{Tl}{12} + d$$

Example 2: Given the formula for electrical power, $P = I^2R$, find the value of P in watts when $I = 15.4$ amperes and $R = 25.7$ ohms.

$$P = I^2R$$

Substituting the given numerical values for I and R:

$$P = (15.4)^2 \times 25.7$$
$$= 237.16 \times 25.7$$
$$= 6{,}095 \text{ watts}$$

Example 3: Given the formula for the total resistance of two resistors in parallel,

$R_T = \frac{R_1R_2}{R_1 + R_2}$, solve for R_T in ohms when

$R_1 = 40$ ohms and $R_2 = 60$ ohms.

$$R_T = \frac{R_1 R_2}{R_1 + R_2}$$

Substitute the given numerical values for R_1 and R_2:

$$R_T = \frac{40 \times 60}{40 + 60}$$

$$= \frac{2{,}400}{100}$$

$$= 24 \text{ ohms}$$

86. Review Problems—Equations

a. Solve for the unknown quantity in each of the following:

(1) $y + 12 = 15$

(2) $\frac{n}{8} = \frac{1}{4}$

(3) $0.63s = 53.55$

(4) $47x - 17 = 235 - 37x$

(5) $(10m + 6) - (11 - 15m) = 14m + 6m$

(6) $x + y = 3$
$3x + 2y = 1$

(7) $a - 3b = 0$
$5a - 4b = 11$

(8) $7x - 5y = 1$
$5x + y = 19$

(9) $4m - 2n = 2$
$3m + n = 14$

(10) $3r - 9s = 15$
$6r - 7s = 41$

b. Solve the following formulas for the quantity indicated:

(1) $Fd = Wh$ for d

(2) $v^2 = v_0^2 + 2gh$ for g

(3) $F = \frac{w}{g} a$ for a

(4) $H = \frac{D^2 N}{2.534}$ for N

(5) $F = \frac{22.5\, BIl}{10^8}$ for l

c. Solve the following linear equations for the unknown quantity:

(1) $7(2x - 6) - 8 = 10x + 10$

(2) $10(x - 2) - 10(2 - x) = 4x - 40$

(3) $9.8a - 9.4 = 6.8a + .6$

(4) $2x + 3 + \frac{11x - 11}{3} = 22$

(5) $3R + (2R - 4) = 6R - 10(R - 2)$

(6) $\frac{5Z}{4} + 2Z = \frac{3 + Z}{3} - 7Z$

(7) $-(5x + 15) = 5x + 21 - \frac{5(2 - x)}{2}$

(8) $\frac{11y - 13}{25} + \frac{17y + 4}{21} + \frac{19y + 3}{7} = 28\frac{1}{7} + \frac{5y - 25\frac{1}{3}}{4}$

(9) $\frac{4X_L}{5} - 6X_L + 2 = \frac{X_L}{4}$

(10) $(x - 1)(x + 1) + x(1 - x) = 4x(2x + 1) - 8x(x - 2)$

d. Solve the following sets of simultaneous linear equations:

(1) $5x - 2y = 10$
$3x - y = 7$

(2) $6a + 15b = 69$
$6a - 6b = 14$

(3) $x - 3y = -17$
$2x + 6y = 50$

(4) $6x - 8y = 20$
$3x + 2y = -14$

(5) $-4x + y = 13$
$8x - 5y = -29$

(6) $2I + \frac{2Z - 22}{3} = 30$
$\frac{3I - 15}{4} + 6Z = 108$

(7) $\frac{2}{x} + y = 1$
$\frac{1}{x} + 2y = 1\frac{1}{4}$

(8) $\frac{a}{3} + \frac{b}{4} = 1$
$\frac{a}{5} + \frac{b}{2} = -\frac{4}{5}$

(9) $\frac{5}{x} + \frac{2}{y} = -1$
$\frac{3}{x} + \frac{1}{y} = 1\frac{3}{5}$

(10) Solve for r and s:
$(a - b)r + (a + b)s = a^2 - b^2$
$(a + b)r - (a - b)s = 2ab$

e. Solve the following problems:

(1) Three times a voltage (E) diminished by 2 is equal to that voltage. What is the voltage?

(2) The sum of two resistances in series is R ohms. One resistance is 20 ohms. Give the algebraic expression for the other.

(3) If a certain voltage (E) is tripled and the result is diminished by 220 volts, the remainder is equal to the original voltage. What is the voltage?

(4) When two resistors are connected in series, the total resistance (R) is the sum of the two resistances. If one resistor is 25 ohms and the total resistance is 100 ohms, what is the value of the other resistor?

(5) The current (I) from a battery is divided among three circuits. The first circuit draws 20 milliamperes more than the second circuit, and the second circuit draws 20 milliamperes more than the third circuit. If the total current drawn is 240 milliamperes, what is the current in each circuit?

(6) Solving by the formula $I = \frac{E}{R}$, how much current (I) does an electric circuit having a resistance (R) of 20 ohms take if the voltage (E) is 110 volts?

Section IX. QUADRATIC EQUATIONS

87. General

A quadratic equation is one which can be reduced to the form $ax^2 + bx + c = 0$ where a, b, and c are known and x is the unknown quantity. In other words, a quadratic equation contains the square of the unknown quantity, such as x^2, but no higher power. For example, $3x^2 + 5x - 2 = 0$ and $x^2 - 4x + 3 = 0$ are quadratic equations. The form $ax^2 + bx + c = 0$ is called the *general quadratic equation.*

88. Pure Quadratic Equations

A pure quadratic equation is obtained from the general quadratic equation when b is equal to zero and the middle term (bx) does not appear. The equation then becomes $ax^2 + c = 0$. The pure quadratic equation has two roots that are equal in absolute value but have opposite signs. As discussed in paragraph 49, all numbers have two square roots. The equation $x^2 - 36 = 0$ is a pure quadratic equation since there are two numbers which, when substituted for x, will satisfy the equation. Thus $(+6)^2 - 36 = 0$ since $36 - 36 = 0$; also, $(-6)^2 - 36 = 0$ since $36 - 36 = 0$. Therefore, $x = \pm 6$.

Example: Solve the equation $x^2 - 5 = 20$ for x.

$$x^2 - 5 = 20$$
$$x^2 = 25$$
$$x = \pm 5$$

Check:

$$(\pm 5)^2 - 5 = 20$$
$$25 - 5 = 20$$
$$20 = 20$$

89. Solution by Factoring

a. Quadratic equations are found in many applications of even the simplest nature. For example, suppose that a sheet of metal is to be cut so that it has an area of 30 square inches, and that the length of the piece will be 1 inch longer than the width. With x representing the unknown width and $x + 1$ the unknown length, $x(x + 1)$ equals the area; therefore, the equation that must be satisfied is $x(x + 1) = 30$. By performing the indicated multiplication and subtracting 30 from each side, the equation now can be written in the form of a quadratic equation, as $x^2 + x - 30 = 0$.

b. To solve this equation, factor the left-hand side into the equivalent equation: $(x - 5)(x + 6) = 0$. The product of two factors is zero if either of the factors is zero (par. 53). Thus, each factor is set equal to zero and solved for the unknown. The equation is satisfied if $x - 5 = 0$ or $x = 5$. Note that the equation also is satisfied if $x + 6 = 0$. This illustrates an important fact concerning quadratic equations: *Every quadratic equation has two solutions.* Only one solution, however, may be appropriate when quadratic equations are used to solve actual problems. The quadratic equation only gives two *possible* solutions—the *actual* solution must be determined by referring to the facts in the original problem.

Example 1: Solve the equation $x^2 - 2x = 0$ for x.

$$x^2 - 2x = 0$$

Factoring:

$$x(x - 2) = 0$$

$$x = 0$$

or $\quad x - 2 = 0$

$$x = 2$$

Thus, 0 or 2 are the roots of the equation $x^2 - 2x = 0$.

Example 2: Solve the equation $2x^2 - 3x - 5 = 0$ for x.

$$2x^2 - 3x - 5 = 0$$

Factoring:

$$(2x - 5)(x + 1) = 0$$

so $\quad x + 1 = 0$

and $\quad x = -1$

or $\quad 2x - 5 = 0$

$$2x = 5$$

and $\quad x = \frac{r}{2}$ or $2\frac{1}{2}$

Thus, -1 and $2\frac{1}{2}$ are the roots of the equation $2x^2 - 3x - 5 = 0$.

90. Solution by Completing the Square

In solving quadratic equations, the method of factoring described in paragraph 89 usually is best if the factors are immediately apparent by inspection. When the values of the unknown are not whole numbers or rational fractions, a quadratic equation can be solved more easily by the method of *completing the square.* This method also is used to derive the quadratic formula (par. 91). For example, to solve the equation $2x^2 - x - 2 = 0$ by completing the square, proceed as follows:

a. Transpose all terms involving x to the left-hand side of the equation and all other terms to the right-hand side. The equation is now in the form $2x^2 - x = 2$, or $x^2 - \frac{1}{2}x = 1$. When using this method, the coefficient of the squared term must be unity (one).

b. Add a number to both sides of the equation so that the left-hand side will be a perfect trinomial square. To determine this number, divide the coefficient of the middle term $(-\frac{1}{2})$ by 2 and square the resulting number.

$$x^2 - \frac{1}{2}x = 1$$

$$x^2 - \frac{1}{2}x + \frac{1}{16} = 1 + \frac{1}{16}$$

c. Replace the trinomial square on the left-hand side of the equation with the square of a binomial.

$$(x - \frac{1}{4})^2 = \frac{17}{16}$$

d. Extract the square root of both sides of the equation.

$$x - \frac{1}{4} = \frac{\pm\sqrt{17}}{4}$$

Thus, $\quad x = \frac{1 \pm \sqrt{17}}{4}$

91. The General Quadratic Equation

a. General. Another method of solving quadratic equations consists of substitution in a formula derived from the general quadratic equation (*b* below). The general quadratic equation is in the form $ax^2 + bx + c = 0$, and any quadratic equation can be written in this form (par. 87). Thus, in the equation $2r^2 + 5r - 3 = 0$, $a = 2$, $b = 5$, and $c = -3$. Similarly, in the equation $9x^2 - 25 = 0$, $a = 9$, $b = 0$, and $c = -25$.

b. Deriving Formula for Solving any Quadratic Equation. Since the general quadratic equation, $ax^2 + bx + c = 0$, represents any quadratic equation, the roots of this equation will represent the roots of any quadratic equation; then, if the general quadratic equation is solved for the unknown values, the roots obtained will serve as a formula for finding the roots of any quadratic equation. The formula is derived from the general form by the method of completing the square; thus, given the general equation $ax^2 + bx + c = 0$, proceed as follows:

(1) Divide through by the coefficient a.

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0$$

(2) Subtract the term $\frac{c}{a}$ from both sides of the equation.

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

This operation prepares the equation for the addition of a quantity to both sides of the equation that will make the left-hand side a perfect square. This quantity is obtained by dividing the coefficient of the x term by 2, and squaring the quotient. Since the coefficient of the x term is $\frac{b}{a}$, the quantity to be added to both sides of the equation is $(\frac{b}{2a})^2$, or $\frac{b^2}{4a^2}$.

(3) Add $\frac{b^2}{4a^2}$ to both sides of the equation.

$$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

(4) Factor the left-hand side of the equation, and add the fraction on the right-hand side.

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

(5) Take the square root of both sides of the equation.

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

(6) Subtract $\frac{b}{2a}$ from both sides of the equation.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

(7) Collect the terms on the right-hand side of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation is known as the ***quadratic formula***. The two roots of any quadratic equation can be obtained by substituting in the formula the particular values of a, b, and c.

92. Solution by the Quadratic Formula

In practical problems, pure quadratic equations (par. 88) are seldom found, and solution by factoring (par. 89) can be used only occasionally. However, any quadratic equation can be solved by the method of completing the square (par. 90)—the method used to derive the quadratic formula (par. 91). This method is unnecessary, however, when the values for a, b, and c for any quadratic equation can be substituted in the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 1: Solve the equation $2x^2 - 6x + 3 = 0$ by using the quadratic formula.

$$2x^2 - 6x + 3 = 0$$

$$a = 2;\ b = -6;\ c = 3$$

Substituting in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{36 - (4)(2)(3)}}{4}$$

$$= \frac{6 \pm \sqrt{12}}{4}$$

$$= \frac{3 \pm \sqrt{3}}{2}$$

Thus, $x = \frac{3 + \sqrt{3}}{2}$ or $x = \frac{3 - \sqrt{3}}{2}$.

Check: $x = \frac{3 + \sqrt{3}}{2}$

$$x = \frac{3 + 1.732}{2} = 2.366$$

Substituting in the equation:

$$2(2.366)^2 - 6(2.366) + 3 = 0$$

$$11.20 - 14.20 + 3 = 0$$

$$14.20 - 14.20 = 0$$

$$x = \frac{3 - \sqrt{3}}{2}$$

$$x = \frac{3 - 1.732}{2} = .634$$

Substituting in the equation:

$$2(.634)^2 - 6(.634) + 3 = 0$$

$$2(.40) - 3.80 + 3 = 0$$

$$3.80 - 3.80 = 0$$

Example 2: Solve the equation $3x^2 + 5x - 2 = 0$ by using the quadratic formula.

$$3x^2 + 5x - 2 = 0$$
$$a = 3;\ b = 5;\ c = -2$$

Substituting in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-5 \pm \sqrt{25 - (4)(3)(-2)}}{(2)(3)}$$
$$= \frac{-5 \pm 7}{6}$$

Thus, $x = \frac{1}{3}$ or $x = -2$.

Check: $x = \frac{1}{3}$

Substituting in the equation:

$$3\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) - 2 = 0$$
$$\frac{3}{9} + \frac{5}{3} - 2 = 0$$
$$\frac{1}{3} + \frac{5}{3} - 2 = 0$$
$$\frac{1}{3} + \frac{5}{3} - \frac{6}{3} = 0$$
$$\frac{6}{3} - \frac{6}{3} = 0$$

$x = -2$

Substituting in the equation:

$$3(-2)^2 + 5(-2) - 2 = 0$$
$$12 - 10 - 2 = 0$$
$$12 - 12 = 0$$

93. Character of the Roots

a. The values for unknowns that are not whole numbers or rational fractions are called *irrational roots*. A *rational* number is a number which can be expressed as the ratio of two integers. For example, 9, $\frac{7}{3}$, $\frac{1}{8}$, and $\sqrt{16}$ are rational numbers. Any whole number is rational since it is the quotient of itself and unity; thus, $9 = \frac{9}{1}$. Numbers such as $\frac{7}{3}$ and $\frac{1}{8}$ are often referred to as rational fractions. A radical is rational if it can be expressed as the quotient of two whole numbers. Thus $\sqrt{16}$ is rational since $\sqrt{16} = 4 = \frac{4}{1}$. A number such as $\sqrt{3}$ which cannot be written as the ratio of two whole numbers is called irrational. Rational and irrational numbers, taken together, make up the system of real numbers. Any number, such as $3 + \sqrt{3}$, which contains a radical sign that cannot be removed also is considered irrational. Roots of quadratic equations are real if a minus sign does not occur under a radical. For example, $x = 5$ is a real root—roots such as $x = \frac{3 + \sqrt{3}}{2}$ or $x = \frac{3 - \sqrt{3}}{2}$ are real, but irrational.

b. One important fact to be remembered when using the quadratic formula is that the expression under the radical sign, $b^2 - 4ac$, must be regarded as a whole before the square root can be taken. The quantity $b^2 - 4ac$ is called the *discriminant* of the quadratic equation. Many things can be learned about a quadratic equation merely by inspecting the discriminant. If the value of the discriminant is positive, real roots will be obtained when the equation is solved. These roots are either rational or irrational—rational when the discriminant is a perfect square, irrational when it is not. The roots are equal only when the value of $b^2 - 4ac$ is zero. When $b^2 - 4ac$ is negative, the square root will be that of a negative number and the roots will be imaginary.

c. In summary, *a quadratic equation always has two solutions.* The solutions will be:

Real and equal .... if $b^2 - 4ac$ equals 0.
Unequal but real ...... if $b^2 - 4ac$ is positive.
Real and rational ..... if $b^2 - 4ac$ is a perfect square.
Imaginary .... if $b^2 - 4ac$ is negative.

94. Review Problems—Quadratic Equations

a. Solve by factoring.

(1) $2x^2 + 3x = 0$
(2) $(x - 4)x = 0$
(3) $(x + 3)\frac{x}{3} = 0$
(4) $\frac{1}{2}x^2 + \frac{1}{3}x = 0$
(5) $2x^2 - 128 = 0$
(6) $\frac{1}{3}x^2 - 2 = 1$
(7) $3x^2 - 25 = 2$
(8) $3x(x - 2) + 2x(3 - x) = 16$
(9) $x^2 - x - 42 = 0$
(10) $x^2 - 13x + 12 = 0$

b. Solve by completing the square.

(1) $x^2 + 3x - 1 = 0$
(2) $y^2 + 6y - 10 = 0$
(3) $E^2 - 4E + 1 = 0$
(4) $2E^2 + 8E - 3 = 0$
(5) $8H^2 - 8H = 5$
(6) $5L^2 - 5 = 2L^2 - 10L$
(7) $14r^2 - 28r - 42 = 0$
(8) $\frac{1}{v^2} - \frac{4}{v} = 2$
(9) $y^2 - 5 = 2y$
(10) $8x^2 - 8x = 8$

c. Solve by using the quadratic formula.

(1) $a^2 + 2a + 1 = 0$
(2) $12y^2 - 6 + y = 0$
(3) $0 = 1 + 5E + 3E^2$
(4) $6I^2 + I - 12 = 0$
(5) $2c^2 + 4c - 6 = 0$
(6) $15R^2 = 22R + 5$
(7) $\frac{Z - 2}{Z} = 1 - Z$
(8) $\frac{3}{r - 2} = 1 + \frac{2}{r + 3}$
(9) $\frac{3x + 2}{2x + 4} = \frac{x + 2}{2x}$
(10) $0 = 6 - \frac{b - 2}{b + 2} + \frac{b - 1}{b + 1}$

CHAPTER 6
GRAPHS

Section I. BASIC CHARACTERISTICS OF GRAPHS

95. General

A graph is a pictorial representation of the relation between two or more quantities. In many instances, problems are more clearly understood when solved graphically than when solved by other methods. Numerical data taken from an experiment or calculations derived from a formula require interpretation, and a curve on a graph depicting such data will provide a picture that shows at a glance how one factor or function depends on another.

96. The Number Line

a. In figure 15, on a straight line of indeterminate length, a point 0 has been chosen from which to measure distances. The point 0 is called the origin. A unit of measurement also has been chosen, and positive and negative integers have been marked off and labeled. The usual choice for a positive direction is shown by the arrow. On the number line, Z_1 corresponds to —4, Z_2 corresponds to 3½, and Z_3 corresponds to 5.2.

b. Consider a number x as corresponding to a point a distance of x units from 0. If x is positive, the point will be in the direction of the arrow from 0; if x is negative, the point will be in the opposite direction from 0. The relative size of two numbers is indicated graphically by the relative positions on the number line of points corresponding to the two numbers. For example, if x is greater than w, the point corresponding to x will be to the right of the point corresponding to w; if x is less than w, the point corresponding to x will be to the left of the point corresponding to w. The number of units from the origin to the point representing a certain number, regardless of direction, is the absolute value (par. 35) of the number.

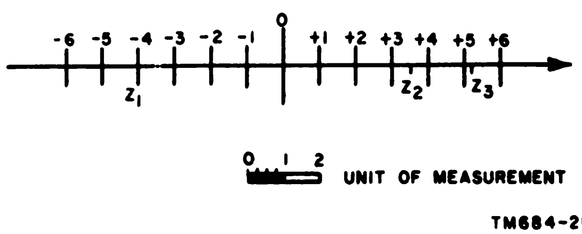


Figure 15. The number line.

97. Rectangular Coordinates

a. In the preceding paragraph, a relationship was given between numbers and points on a straight line. A similar relationship can be established between a pair of numbers and a point on a plane. In figure 16, two number lines are drawn perpendicular to each other at their origins for form a set of axes. The horizontal axis is commonly called the *x axis;*

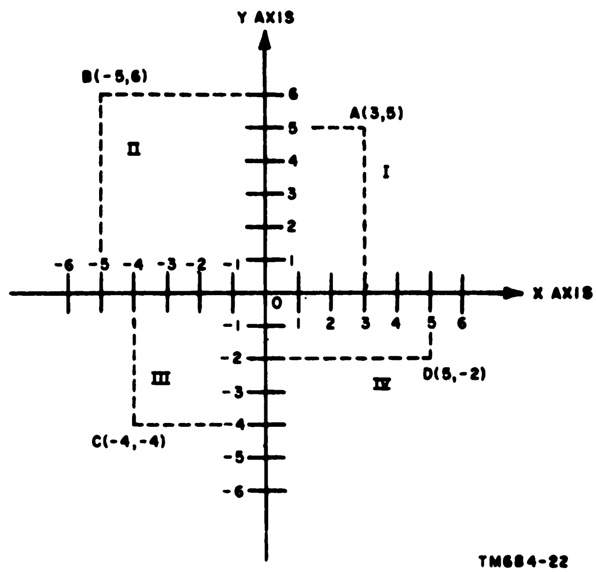


Figure 16. Rectangular coordinates.

the vertical axis is commonly called the *y axis.* Any point on the plane can be located with reference to the two axes: It must lie a certain number of units to the left (negative) or to the right (positive) of the y axis; and it must lie a certain number of units above (positive) or below (negative) the x-axis. To locate a point with reference to the set of axes, it is necessary only to know the x value and the y value of the point. These two values are known as the *coordinates* of the point. The x value, called the *abscissa,* is written first; the y value, called the *ordinate,* follows. The two numbers are separated by a comma and are usually inclosed in parentheses. Thus, in figure 16, the correct notation for the coordinates at point A is (3,5), because the x value is 3 and the y value 5.

b. The axes divide the graph into four sections, or *quadrants,* identified by the Roman numerals I, II, III, and IV in figure 16. The signs of the abscissa and the ordinate in each of the quadrants are given in the chart below.

Quadrant	Abscissa	Ordinate
I	+	+
II	—	+
III	—	—
IV	+	—

98. Plotting Points

The procedure for locating points by their coordinates is called *plotting* the points. To plot the point D (5, —2) in figure 16, for example, erect a perpendicular on the x axis five units to the right of the y axis; then erect a perpendicular to the y axis two units below the x axis; the point of intersection of these two perpendiculars is the point D (5,—2).

99. Review Problems—Plotting Points

a. Plot each of the following points and state the quadrant, if any, in which each lies:

(1) (4,2)
(2) (4,—2)
(3) (—1,3)
(4) (6,—1)
(5) (3,0)
(6) (0,—3)
(7) (—15,—27)
(8) ($3\frac{1}{2}$,$4\frac{3}{5}$)
(9) (5.6,—6.5)

b. Plot the points in the following chart and connect them by straight segments in the order of increasing values of x:

x	—3	—2	—1	0	1	2	3	4
y	18	8	2	0	2	8	18	32

c. Plot the points in the following chart and sketch a smooth curve passing through them in the order of increasing values of x:

x	—3	—2	—1	0	1	2	3
y	—37	—8	5	8	7	7	17

d. If $y = 2x - 3$, plot the points for which x = 4, 2, 1, 0, —1, —2, and —4 after finding the corresponding values of y.

e. Draw the triangle of which the vertices are (—2,6), (3,2), and (0,—3).

f. Draw the quadrilateral of which the vertices, connected in the order given, are (1,3), (—3,4), (—2,—5), and (3,—2).

Section II. GRAPHING EQUATIONS

100. Graphing Linear Equations

a. General. An equation in the first degree in two unknowns is called a *linear equation* since its graph is a straight line. For example, $x + y = 5$, $2x + y = 12$, and $x - 6y = 6$ are linear equations. An equation is said to be of the first degree in two unknowns if only the first power of either unknown is involved and if neither of the unknowns appears in a denominator.

b. Plotting Graphs of Linear Equations.

(1) The first step in plotting the graph of a linear equation (or of any other equation or formula) is to set up a table of values for both unknowns that will satisfy the equation. In the equa-

tion $x + y = 5$, for example, it is apparent that there are a number of values for x and y that will satisfy the equation. For any number assigned to x, there is a corresponding number for y which will satisfy the equation. Consider that 4 and —4 will be the maximum plus and minus values for x. Using the values 4, 3, 2, 1, 0, —1, —2, —3, and —4 for x, the equation is solved for y at each value of x. These are arranged in tabular form as shown on figure 17.

(2) Each of these pairs of values gives a point on a graph. Consider each of the corresponding points as coordinates—the value of x the abscissa and the value of y the ordinate. The line joining these points (fig. 17) is the graph of the equation $x + y = 5$. Note that the coordinates for any two points are sufficient to determine its graph. Therefore, plotting the coordinates for any two points is sufficient to determine the graph of a first degree equation. Plotting a third point, however, will serve as a check, for if the three points are not on the same straight line, one of them is in error.

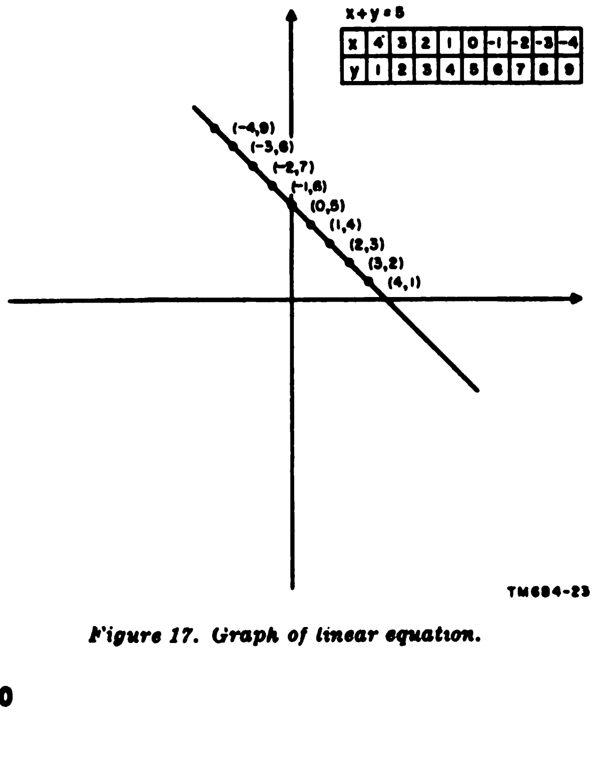


Figure 17. Graph of linear equation.

101. Graphical Solution of Simultaneous Linear Equations

a. When two *independent* linear equations contain the same two related unknowns, there will be an unlimited number of solutions for each equation. However, *there can be only one set of values that will satisfy both equations.* Determining the one set of values is known as the simultaneous solution of the two independent equations.

b. Graphically, the two equations can be solved simultaneously by plotting them on the same graph and locating their point of intersection (if there is one). For example, consider the graphical solution of the equations $3x — 2y = 0$ and $3x + 2y = 6$. Selecting 6 and —6 as the maximum plus and minus values for x and using $x = 4$ as a checkpoint, the coordinates for both equations are determined. For the equation $3x — 2y = 0$, these coordinates are (6,9), (4,6), and (—6,—9); for the equation $3x + 2y = 6$, (6,—6), (4,—3), and (—6,12). These coordinates are plotted on an axis and a line is drawn joining the plotted points of each equation (fig. 18). The graphs of the two independent linear equation cross at point P, where $x = 1$ and $y = 1.5$. To check the graphical solution of the equations, substitute these values for x and y in the original equations. Since they satisfy both equation, the graphical solution is correct.

c. If two *dependent* equations are plotted on a graph, their lines will coincide. For example, the equations $x + y = 4$ and $2x + 2y = 8$

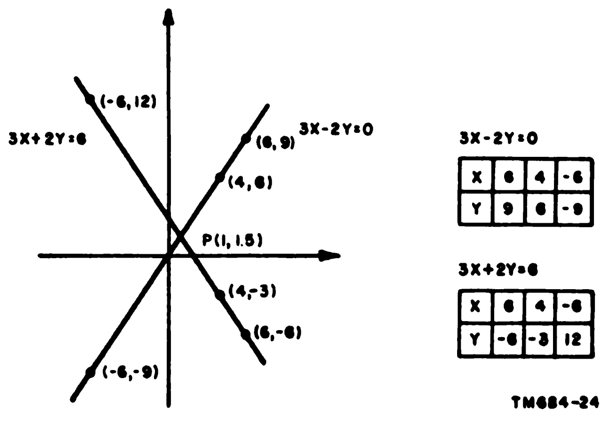


Figure 18. Graphical solution of simultaneous linear equations.

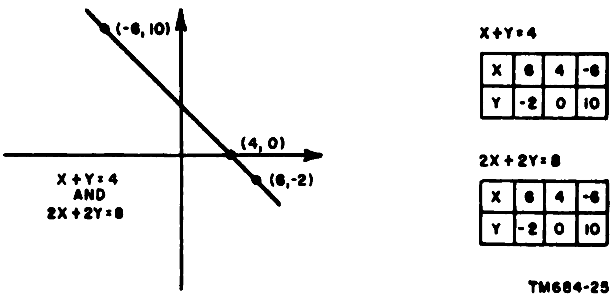


Figure 19. Graph of dependent simultaneous linear equations.

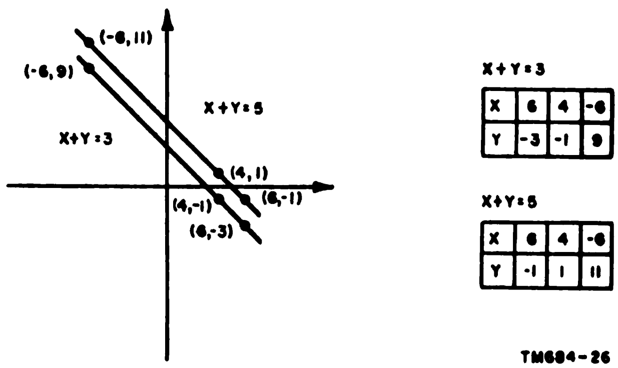


Figure 20. Graph of inconsistent simultaneous linear equations.

are dependent, since they can be reduced to identical forms. Selecting the same plus and minus values for x and the same checkpoint as in *b* above, the coordinates for both equations are found to be (6,—2), (4,0), and (—6,10). Plotted on a graph, both equations form a single line (fig. 19).

d. Simultaneous equations that have no common solution are called *inconsistent.* No solution is possible for the equations $x + y = 3$ and $x + y = 5$, because there are no values for x and y which, when added together to make 3, will also equal 5. Using 6 and —6 as maximum plus and minus values for x, and using $x = 4$ as a checkpoint, the coordinates for equation $x + y = 3$ are found to be (6,—3), (4,—1), and (—6,9); the coordinates for $x + y = 5$ are (6,—1), (4,1), and (—6,11). Plotted on a graph, these equations form parallel lines (fig. 20).

102. Graphing Quadratic Equations

a. The Dependent Variable. In graphing a quadratic equation, only two values, or points, for plotting the equation can be obtained by finding the roots of the equation (par. 88). These values do not give a complete picture of the equation. To get a continuous graph, a *dependent variable* is introduced. This variable, usually identified by the letter y, gets its name from the fact that it depends on another quantity for its value. For example, in the equation $y = x^2 - 6x + 5$, the value of y depends on the value of x; therefore, y is a dependent variable. The quantity on which y depends is called the *independent variable.* A more accurate designation for the dependent variable is $f(x)$, meaning *function of x.* Using this designation, the equation given above would be written $f(x) = x^2 - 6x + 5$. If the independent variable in the equation were z, the equation would be written $f(z) = z^2 - 6z + 5$.

b. Graphical Solution of Quadratic Equations. In the original equation $f(x) = x^2 - 6x + 5$, different values are substituted for the unknown to find the corresponding values of the function; thus if x equals —1, the equation becomes $f(-1) = (-1)^2 - 6(-1) + 5 = 12$; if x equals zero, the equation becomes $f(0) = 0 - 0 + 5 = 5$; if x equals 1, the equations becomes $f(1) = (1)^2 - 6(1) + 5 = 0$, etc. Compile a table of enough values to make it possible to plot the equation, as shown in figure 21. The graph of the function crosses the x-axis at two points, 1 and 5, which give a graphical solution of the equation $x^2 - 6x + 5 = 0$. The equation also may be solved by factoring, as follows:

$$(x - 1)(x - 5) = 0$$
$$x - 1 = 0 \text{ and } x - 5 = 0$$
$$x = 1 \text{ and } x = 5$$

Thus, the solutions or the roots of the equation are obtained when $f(x) = 0$. These roots represent the points where the graph of $f(x) = x^2 - 6x + 5$ crosses the x-axis.

c. Properties of Functions. In addition to the original equation, $f(x) = x^2 - 6x + 5$, consider three equations that differ in one respect —their constant terms are not the same. For example:

$$f(x) = x^2 - 6x + 8$$
$$f(x) = x^2 - 6x + 9$$
$$f(x) = x^2 - 6x + 12$$

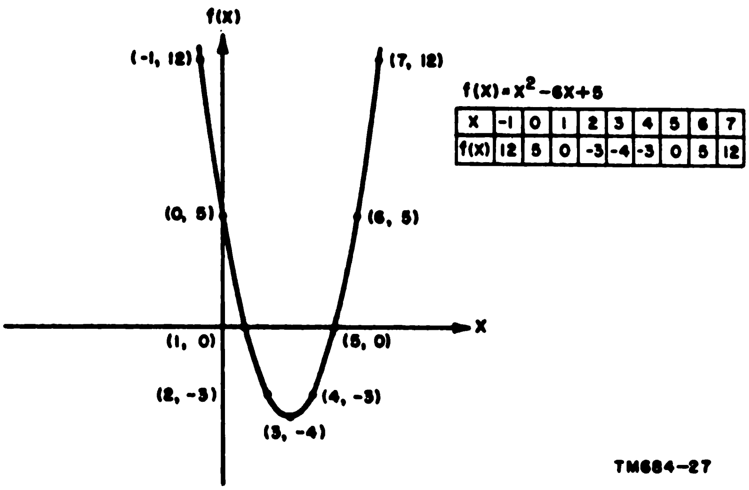


Figure 21. Graph of function of quadratic equation.

The graphs of the four corresponding functions have interesting properties and can be studied more advantageously when plotted on the same graph, as shown in figure 22.

(1) The function of $x^2 - 6x + 5$ crosses the horizontal or x-axis at two points, 1 and 5. These points indicate that the roots of the equation are $x = 1$ and $x = 5$. To compare this information with the discussion on quadratic equations in chapter 5, the discriminant of the equation must be investigated. The discriminate of $x^2 - 6x + 5$ is $(b^2 - 4ac) = (36 - 4 \cdot 1 \cdot 5) = 36 - 20 = 16$. Referring to the summary of the character of roots in paragraph 93, the roots are real and rational. To prove this, substitute the value of the discriminant in the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{16}}{2}$$

$$x = \frac{6+4}{2} = 5 \text{ or } \frac{6-4}{2} = 1$$

Thus, the discriminant is a perfect square and the roots are real and rational.

(2) The function of $x^2 - 6x + 8$ crosses the horizontal axis at 2 and 4, indicating that the roots are $x = 2$ and $x = 4$. Calculating the discriminant, $(b^2 - 4ac) = (36 - 4 \cdot 2 \cdot 2) = 36 - 32 = 4$. Thus, the discriminant is a perfect square and will give real and rational roots.

(3) The function of $x^2 - 6x + 9$ touches the x-axis at only one point, 3. Thus, both roots of the equation are $x = 3$. Calculating the discriminant, $(b^2 - 4ac) = (36 - 4 \cdot 9) = 0$, which indicates that the roots are real and equal. Check the graph of this equation (fig. 22); it will be seen that the curve just touches the x-axis at one point. Thus, the root $x = 3$ must be counted twice and may be called a double root.

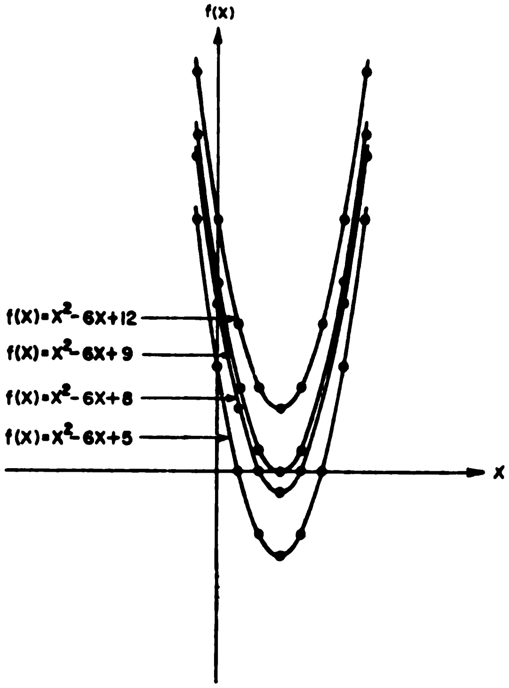
(4) The equation $f(x) - x^2 - 6x + 12$ has a discriminant equal to $(36 - 4 \cdot 12)$ or -12. Solving for the roots of this equation,

$$x = \frac{6 \pm \sqrt{-12}}{2} = 3 \pm \sqrt{-3}.$$

This is imaginary, but the meaning becomes apparent when the graph of the function of the equation is inspected. The plot does not cross the x-axis and, therefore, both roots must be imaginary.

d. Minimum Value of a Quadratic.

(1) The minimum value of a quadratic function will occur at $x = \frac{-b}{2a}$ when



$f(X) = X^2 - 6X + 5$

X	-1	0	1	2	3	4	5	6	7
f(X)	12	5	0	-3	-4	-3	0	5	12

$f(X) = X^2 - 6X + 8$

X	-1	0	1	2	3	4	5	6	7
f(X)	15	8	3	0	-1	0	3	8	15

$f(X) = X^2 - 6X + 9$

X	-1	0	1	2	3	4	5	6	7
f(X)	16	9	4	1	0	-1	4	9	16

$f(X) = X^2 - 6X + 12$

X	-1	0	1	2	3	4	5	6	7
f(X)	19	12	7	4	3	4	7	12	19

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Figure 22. Properties of functions.

the general quadratic equation $ax^2 + bx + c = y$ (par. 91) defines the coefficients a and b. This relation can be checked by calculating the value of x at which the minimum value of the function $x^2 - 6x + 5$ occurs and comparing this calculated value with the plot of the equation (fig. 21 or 22). Thus,

$$x = \frac{-b}{2a} = -\frac{(-6)}{2(1)} = \frac{6}{2} = 3,$$

and the minimum value of the function $x^2 - 6x + 5$ occurs at $x = 3$. Checking the graph verifies this statement. The minimum value of the functions $x^2 - 6x + 8$, $x^2 - 6x + 9$, and $x^2 - 6x + 12$ also occurs at $x = 3$.

(2) To find the value of the function at the minimum point, substitute for x. The minimum occurs at $x = \frac{-b}{2a}$; therefore, substitute $\frac{-b}{2a}$ for x in the function of the general quadratic equation.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c \\ &= \frac{b^2}{4a} - \frac{b^2}{2a} + c = \frac{b^2}{4a} - \frac{2b^2}{4a} + c \\ &= \frac{-b^2}{4a} + c \end{aligned}$$

Thus, to find the value of the function $f(x) = x^2 - 6x + 5$ at the minimum point:

$$f(x) = \frac{-b^2}{4a} + c = \frac{-36}{4} + 5 = -9 + 5 = -4$$

This method can be used to find the minimum value of the function if the value of x at which the minimum occurs is *not* known. However if it is known that the minimum value occurs at $x = 3$, merely substitute this value for x in the original equation.

$$\begin{aligned} f(x) &= x^2 - 6x + 5 \\ &= 9 - 6 \cdot 3 + 5 \\ &= 14 - 18 \\ f(x)\text{min} &= -4 \end{aligned}$$

(3) Note that in all cases where the word *minimum* is used, the word *maximum* is applicable if the equation $y = f(x)$ is such that its graph has a maximum instead of a minimum. If the equation were $f(x) = 3 + 6x - x^2$, the minus sign preceding the term x^2 would indicate that the curve has a maximum.

e. Practical Application. The methods of analysis presented in *c* and *d* above can be used for some very important relationships in applied electricity and electronics. It may be used, for example, to find the load resistance of a circuit in terms of the circuit components necessary to obtain maximum power transfer (par. 216).

103. Review Problems—Graphs

a. Plot the graphs of the following linear equations:

(1) $2x - 5 = y$

(2) $5 - 2x = y$

(3) $y = 5x$

(4) $3x + 2y = 18$

(5) $5x - 5y = 20$

(6) $3x + y + 14 = 0$

b. Plot the graphs of the following sets of simultaneous equations:

(1) $2x + 3y = 12$
$3x - y = 7$

(2) $x + y = 9$
$5x + y = 17$

(3) $x + 5y = 22$
$3x - 2y = -2$

(4) $3x - 2y = 0$
$x - 5y = 18$

(5) $6x + 2y = 12$
$4y + 2y = 10$

(6) $x - 2y = 0$
$y = 1 + x$

c. Find the roots of the following quadratic equations to the nearest tenth by plotting their graphs:

(1) $y^2 - 2y - 2 = 0$

(2) $x^2 - 1 + x = 0$

(3) $9 - t^2$

(4) $x^2 - 2x + 2 = 0$

(5) $x^2 - 5x + 3 = 0$

(6) $10 - 3x - x^2 = 0$

CHAPTER 7

POWERS OF 10

104. General

The technique of using powers of 10 can greatly simplify mathematical calculations. A number containing many zeros to the right or to the left of the decimal point can be dealt with much more readily when put in the form of powers of 10. For example, .0000037 × .000021 can be handled more easily when put in the form $3.7 \times 10^{-6} \times 2.1 \times 10^{-5}$.

105. Table of Powers of 10

The table below gives some of the values of the powers of 10. In a whole number, the exponent is positive and equals the number of zeros following the 1; in decimals, the exponent is negative and equals one more than the number of zeros immediately following the decimal point.

Number	Power of 10	Number	Power of 10
.000001	10^{-6}	1	10^{0}
.00001	10^{-5}	10	10^{1}
.0001	10^{-4}	100	10^{2}
.001	10^{-3}	1,000	10^{3}
.01	10^{-2}	10,000	10^{4}
.1	10^{-1}	100,000	10^{5}
		1,000,000	10^{6}

106. Expressing Numbers in Scientific Notation

Any number written as the product of an integral power of 10 and a number between 1 and 10 is said to be expressed in *scientific notation.*

Example 1: $81{,}000{,}000 = 8.1 \times 10{,}000{,}000 = 8.1 \times 10^{7}$

Example 2: $600{,}000{,}000 = 6 \times 100{,}000{,}000 = 6 \times 10^{8}$

Example 3: .000,000,000,9 = 9 × .000,000,-000,1 = 9×10^{-10}

107. Addition and Subtraction of Numbers in Scientific Notation

Numbers expressed in scientific notation can only be added or subtracted if the powers of 10 are the same. For example, 3×10^{5} can be added to 2×10^{5} to get 5×10^{5}; however, 3×10^{6} cannot be added to 2×10^{5} because the powers of 10 are not the same. The number 3×10^{6} can be changed to 30×10^{5}, however, and it can then be added to 2×10^{5} to obtain 32×10^{5}. The answers to problems solved by using scientific notation can be left in the exponential form. In the examples below, however, the answers are converted to the decimal form to aid in understanding this technique.

Example 1: Add 450,000 and 763,000.

$$450{,}000 + 763{,}000 = 45 \times 10^{4} + 76.3 \times 10^{4} = 121.3 \times 10^{4} = 1{,}213{,}000$$

Example 2: Add .000,068,25 and .000,007,54.

$$.000{,}068{,}25 + .000{,}007{,}54 = 6825 \times 10^{-8} + 754 \times 10^{-8} = 7579 \times 10^{-8} = .000{,}075{,}79$$

Example 3: Subtract .000,004,33 from .000,05.

$$.000{,}05 - .000{,}004{,}33 = 5000 \times 10^{-8} - 433 \times 10^{-8} = 4567 \times 10^{-8} = .000{,}045{,}67$$

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108. Multiplication of Numbers in Scientific Notation

The general rules covering the multiplication of radicals (par. 74) also apply in the multiplication of numbers that are expressed in scientific notation.

Example 1: Multiply 100,000 by 1,000.

$$100{,}000 \times 1{,}000 = 10^5 \times 10^3 = 10^{5+3} = 10^8 = 100{,}000{,}000$$

Example 2: Multiply 25,000 by 5,000.

$$\begin{aligned} 25{,}000 \times 5{,}000 &= 2.5 \times 10^4 \times 5 \times 10^3 = 2.5 \times 5 \times 10^{4+3} \\ &= 12.5 \times 10^7 \\ &= 125{,}000{,}000 \end{aligned}$$

Example 3: Multiply 1,800, .000015, 300, and .0048.

$$\begin{aligned} &1{,}800 \times .000015 \times 300 \times .0048 \\ &= 1.3 \times 10^3 \times 1.5 \times 10^{-5} \times 3 \times 10^2 \times 4.8 \times 10^{-3} \\ &= 1.8 \times 1.5 \times 3 \times 4.8 \times 10^{3-5+2-3} \\ &= 38.88 \times 10^{-3} \\ &= .03888 \end{aligned}$$

109. Division of Numbers in Scientific Notation

The general rules covering the division of radicals (par. 75) also apply in the division of numbers that are expressed in scientific notation.

Example 1: Divide 75,000 by .0005.

$$\frac{75{,}000}{.0005} = \frac{75 \times 10^3}{5 \times 10^{-4}} = \frac{75}{5} \times 10^{3+4} = 15 \times 10^7 = 150{,}000{,}000$$

Example 2: Divide 14,400,000 by 1,200,000.

$$\frac{14{,}400{,}000}{1{,}200{,}000} = \frac{144 \times 10^5}{12 \times 10^5} = \frac{144}{12} = 12$$

Example 3: Divide 98,100 by .0025, 180, and 1,090,000.

$$\begin{aligned} &\frac{98{,}100}{.0025 \times 180 \times 1{,}090{,}000} \\ &= \frac{9.81 \times 10^4}{2.5 \times 10^{-3} \times 1.8 \times 10^2 \times 1.09 \times 10^6} \\ &= \frac{9.81 \times 10^4}{2.5 \times 1.8 \times 1.09 \times 10^{-3+2+6}} \\ &= \frac{9.81 \times 10^4}{4.905 \times 10^5} \\ &= 2 \times 10^{-1} \\ &= .2 \end{aligned}$$

110. Finding the Power or Root of a Number in Scientific Notation

The general rules covering powers and roots (pars. 71 and 72) also apply to numbers expressed in scientific notation.

Example 1: Find the square root of 144,000,000.

$$\begin{aligned} \sqrt[2]{144{,}000{,}000} &= \sqrt[2]{144 \times 10^6} \\ &= 12 \times 10^3 \\ &= 12{,}000 \end{aligned}$$

Example 2: **Find the cube root of .000,008.**

$$\sqrt[3]{.000{,}008} = \sqrt[3]{8 \times 10^{-6}}$$
$$= 2 \times 10^{-2}$$
$$= .02$$

Example 3: **Square 15,000.**

$$(15{,}000)^2 = (15 \times 10^3)^2$$
$$= 225 \times 10^6$$
$$= 225{,}000{,}000$$

Example 4: **Find the square root of $(160{,}000)^3$.**

$$\sqrt[2]{160{,}000^3} = (160{,}000)^{3/2}$$
$$= (16 \times 10^4)^{3/2}$$
$$= 64 \times 10^6$$
$$= 64{,}000{,}000$$

Example 5: **Find the square root of $\frac{86{,}900}{3{,}560{,}000}$.**

$$\sqrt{\frac{86{,}900}{3{,}560{,}000}} = \sqrt{\frac{8.69 \times 10^4}{3.56 \times 10^6}}$$
$$= \sqrt{2.44 \times 10^{-2}}$$
$$= 1.56 \times 10^{-1}$$
$$= .156$$

111. Review Problems—Powers of 10

In the following problems, leave the answer in powers of ten:

a. Convert the following numbers to powers of 10 and add:

(1) 1,245,000 + 368,000

(2) 79,000 + 421,000

(3) .000,007,66 + .000,054

b. Convert the following numbers to powers of 10 and subtract:

(1) 333,400 — 22,500

(2) .000,068 — .000,049

(3) .000,004,89 — .000,000,398

c. Convert the following numbers to powers of 10 and multiply:

(1) 446,000 × 200

(2) 7,700 × .003,2

(3) .000,096 × .000,33

(4) .003,66 × 4,000,000

d. Convert the following numbers to powers of 10 and divide:

(1) 668,000 ÷ 4,000

(2) 88.445,000 ÷ .000,55

(3) .000,963 ÷ .000,009

(4) .006,93 ÷ 21

e. Convert the following numbers to powers of 10 and perform the indicated operations:

(1) $\sqrt[3]{64{,}000{,}000}$

(2) $\sqrt[2]{.000{,}169}$

(3) $.003^3$

(4) $27{,}000^{2/3}$

CHAPTER 8
LOGARITHMS

112. General

Many lengthy mathematical operations may be accomplished more easily through the use of logarithms. With logarithms (also called logs), multiplication of numbers is reduced to a simple process of addition, division becomes a process of subtraction, raising a number to a power becomes simple multiplication, and extraction of roots is done by simple division.

113. Definition

The logarithm of a given number is the power to which another number (called the base) must be raised to equal the given number. The word "logarithm" has the same meaning as the word "exponent."

Example: Find the logarithm of 1,000 to the base 10.

From the definition, the logarithm of a number (1,000) is the power (x) to which another number called the base (10) must be raised to equal the given number (1,000).

Thus, $10^x = 1{,}000$. Since $10^3 = 1{,}000$, then:

$10^x = 10^3$ and by inspection:

$x = 3$

Therefore, the logarithm of 1,000 to the base 10 equals 3 or $\log_{10} 1{,}000 = 3$.

114. Types of Logarithms

a. Common Logarithms. Common logarithms use the number 10 as a base. They are so universally used that the 10 usually is omitted; the answer in paragraph 113 could be log 1,000 = 3. Some values of common logarithms are included in the table below. The common logarithm of any number between these values consists of the logarithm of the smaller number plus a decimal. For example, the log of a number between 100 and 1,000, such as 157, consists of the log of the smaller number (10) plus a decimal. The log of 157 is 2.1959.

log 1 = 0	log .1 = —1
log 10 = 1	log .01 = —2
log 100 = 2	log .001 = —3
log 1,000 = 3	log .0001 = —4
log 10,000 = 4	

b. Natural Logarithms. Natural logarithms are based upon the irrational number *e*, and are written both as $\log_e$ and ln. Natural logarithms are used in special applications and as such are not explained further in this text.

115. Parts of Logarithms

a. Logarithms are divided into two parts, the integral and the decimal. The integral part is known as the *characteristic*, and the decimal part is called the *mantissa*.

(1) *The characteristic of any number is one less than the number of digits to the left of the decimal point.* Thus, the characteristic for the number 3 is 1 — 1 or zero, since there is one number to the left of the decimal point. The characteristic for 30, with two numbers to the left of the decimal point, is 2 — 1 or 1. Similarly, the characteristic for 300 is 2, and the characteristic for 3,000 is 3. The characteristic of the log of a decimal is negative and is based upon the position of the first rational number to the right of the decimal point. *If there are no numbers to the left of the decimal point, the characteristic is negative.* In the number .327, for example, the first

rational number is in the first decimal place and the characteristic is —1; in the number .03, the first rational number is in the second decimal place and the characteristic is —2. Similarly, the characteristic for .003 is —3, and the characteristic for .0003 is —4.

(2) The mantissa is always the same for a given sequence of integers, regardless of where the decimal point appears among them. Thus, the *mantissa* is the same for 1570, 157, 15.7, 1.57, .157, and .0157, and the logs of these numbers differ only in respect to their characteristics. Their logarithms, respectively, are 3.1959, 2.1959, 1.1959, 0.1959, —1.1959 and —2.1959.

b. The mantissa is always positive—even when the characteristic is negative. This fact poses a problem of notation, and also complicates the addition and subtraction of logarithms.

(1) In the notation of logarithms, to say that log .157 is —1.1959 is not strictly true, for what we mean to say is —1 plus .1959. To overcome this problem, the minus sign is generally written above the characteristic, and is made long enough to cover the entire negative portion of the logarithm. More properly, therefore, log .157 is written $\bar{1}.1959$.

(2) In the addition and subtraction of logarithms, the complication can be removed by expressing the negative characteristic in a positive manner; more precisely, by adding a large enough number to the characteristic and by subtracting the same number from the entire logarithm. Thus, the log of .157 is written 9.1959-10, and the log of .0157 is written 8.1959—10.

116. Finding a Logarithm

A table of common logarithms is given in appendix III. Note, however, that the table contains only the mantissas of logarithms. The characteristic must be obtained, in each instance, by following the rules given in paragraph 115*a*(1).

Example 1: Find the logarithm of 333.

Determine the characteristic of 333. The characteristic is 3 —1, or 2.

Determine the mantissa of 333. In the table of common logarithms, look down the N column for the number 33. The mantissa for 333 is in this horizontal row in the column headed by the number 3. The mantissa is .5224.

Log 333 = 2.5224.

Example 2: Find the logarithm of .127.

Determine the characteristic of .127. The characteristic is —1 or 9. -------- —10.

Determine the mantissa of .127. In the table of common logarithms, look down the N column for 12. The mantissa for 127 is in this horizontal row in the column headed by the number 7. The mantissa is .1038.

Log .127 = 9.1038—10.

117. Logarithmic Interpolation

The table of common logarithms given in appendix III is adequate if the given number has three or less integers. If it has four or more integers, however, it is necessary to interpolate—that is, to find the proportional part of the difference between the logarithms shown in the table.

Example 1: Find the logarithm of 2.369.

Step 1. The characteristic of 2.369 is 0. Since the mantissa for this number cannot be found in the table, it is necessary to interpolate. Look for the mantissas of the numbers next lower and higher than 2369. The mantissa of the number 2360 is .3729 and the mantissa of the number 2370 is .3747. Since 2369 lies between 2360 and 2370, the mantissa of

2369 must lie between .3729 and .3747. This may be written:

$$\log 2360 = .3729$$
$$\log 2369 = .3729 + x$$
$$\log 2370 = .3747$$

Step 2. Set up the proportions. The difference between 2369 and 2360 is 9. The difference between 2370 and 2360 is 10. Therefore, the desired mantissa is $\frac{9}{10}$ of the difference between these two. Let the difference between the mantissa of 2369 and 2360 equal x. The difference between .3747 and .3729 is .0018. The proportion is $\frac{x}{.0018}$.

Step 3. Solve the problem.

$$\frac{9}{10} = \frac{x}{.0018}$$
$$10x = .0162$$
$$x = .0016$$

Step 4. Since the value of x is .0016, the mantissa of 2369 is .3729 + .0016 or .3745. Therefore, log 2.369 = 0.3745.

Example 2: Find the logarithm of .017234.

Step 1. The characteristic of .017234 is —2 or 8. ________ —10. The numbers in the table lower and higher than 17234 are 17200 and 17300. The mantissa of 17200 is .2355; the mantissa of 17300 is .2380. The difference between 17234 and 17200 is 34; the difference between 17300 and 17200 is 100; the difference between .2380 and .2355 is .0025. This may be written:

$$\log 17200 = .2355$$
$$\log 17234 = .2355 + x$$
$$\log 17300 = .2380$$

Step 2. Let the difference between the mantissas of 17234 and 17200 equal x. The equation is as follows:

$$\frac{34}{100} = \frac{x}{.0025}$$
$$100x = .0850$$
$$x = .00085 = .0009$$

Step 3. Since the value of x is .0009, the mantissa of 17234 is .2355 + .0009 or .2364. Therefore, log .017234 = 8.2364—10.

118. Reading Antilogarithms

The process of finding the antilogarithm (also called antilog), consists of determining the number from which the logarithm was derived. This process is essentially the reverse of finding the logarithm (par. 116). Consequently, the location of the decimal point is determined from the characteristic, and the numerical value of the number is determined from the mantissa.

Example 1: Find the antilog of 1.8954.

Step 1. Since the characteristic of the logarithm is 1, there will be two digits to the left of the decimal point in the number.

Step 2. Look in the table for the mantissa, .8954. The number given for .8954 is 786.

Step 3. Count off two digits from the left and insert the decimal point. The antilog of 1.8954 is 78.6.

Example 2: Find the antilog of 7.0828—10.

Step 1. Since the characteristic of the logarithm is —3, the first significant figure will be in the third decimal place.

Step 2. Look for the mantissa .0828 in the table. The number given for .0828 is 121.

Step 3. Add two zeros to the right of the decimal point and before the first significant figure. Thus, the antilog of 7.0828—10 is .0021.

119. Antilogarithmic Interpolation

If the mantissa of a logarithm does not appear in the table, it is necessary to interpolate.

Example 1: Find the antilog of 2.7654.

Step 1. Since the characteristic of the logarithm is 2, there will be three digits to the left of the decimal point in the number.

Step 2. The mantissa in the table lower than .7654 is .7649. The number with .7649 as a mantissa is 582.

Step 3. The mantissa higher than .7654 is .7657. The number with .7657 as a mantissa is 583.

Step 4. Set up the proportions. The difference between .7654 and .7649 is .0005; the difference between .7657 and .7649 is .0008. The proportional difference is $\frac{.0005}{.0008}$ or $\frac{5}{8}$. The difference between 583 and 582 is 1. This can be written:

antilog .7649 = 582
antilog .7654 = 582 + x
antilog .7657 = 583

Step 5. Let x equal the difference between the number represented by the mantissa .7654 and the number 582. The equation is as follows:

$$\frac{5}{8} = \frac{x}{1}$$
$$8x = 5$$
$$x = .625$$

Step 6. The number is 582 + .625. Since there are three digits to the left of the decimal point, the antilog of 2.7654 is 582.625.

Example 2: Find the antilog of 6.7166—10.

Step 1. Since the characteristic of the logarithm is —4, the first rational number will be in the fourth decimal place.

Step 2. The mantissa in the table lower than .8166 is .8162; the number with .8162 as a mnatissa is 655.

Step 3. The mantissa in the table higher than .8166 is .8169; the number with .8169 as a mantissa is 656.

Step 4. The difference between .8162 and .8166 is .0004; the difference between .8169 and .8162 is .0007. The proportional difference is $\frac{.0004}{.0007}$ or $\frac{4}{7}$. The difference between 656 and 655 is 1. This may be written:

antilog .8162 = 655
antilog .8166 = 655 + x
antilog .8169 = 656

Step 5. Let x equal the difference between the number represented by the mantissa .8166 and the number 655. The equation is as follows:

$$\frac{4}{7} = \frac{x}{1}$$
$$7x = 4$$
$$x = .57$$

Step 6. The number is 655 + .57. Since the first rational figure is in the fourth decimal place, the antilog of 6.7166—10 is .00065557.

120. Addition and Subtraction of Logarithms

Logarithms are added and subtracted arithmetically. Since every mantissa is positive (par. 115*b*), however, every negative characteristic should be expressed as a posiitve (par. 115*b*).

Example 1: Add the logarithms 3.7493 and 2.4036.

```
  3.7493
+2.4036
  6.1529
```

Example 2: Add the logarithms 3.4287 and $\bar{6}$.3982.

```
  3.4287
+4.3982—10
  7.8269—10
```

Example 3: Add the logarithms 8.9324—10, 7.2812—10, 5.4138—10, and 9.9918—10.

```
    8.9324—10
    7.2812—10
    5.4138—10
+   9.9918—10
   31.6192—40
—(30      —30)
    1.6192—10
```

Example 4: Subtract the logarithm 9.1245 from the logarithm 6.3058.

To subtract a larger logarithm from a smaller logarithm, add 10 or a multiple of 10 to the smaller logarithm, and subtract the same number from the logarithm by writing that number with a minus sign to the right of the logarithm. The number chosen for this purpose should be the least that will cause the smaller logarithm to exceed the larger.

$$\begin{array}{r} 16.3058\text{—}10 \\ -\ 9.1245 \\ \hline 7.1813\text{—}10 \end{array}$$

Example 5: Subtract the logarithm 3.7980—10 from 2.8686. When subtracting a negative logarithm from a positive logarithm, where that part of the characteristic of the negative logarithm to the left of the mantissa is larger than the characteristic of the positive logarithm, add 10 or a multiple of 10 to the characteristic of the positive logarithm, and subtract that same amount from the right of the positive logarithm.

$$\begin{array}{r} 12.8686\text{—}10 \\ 3.7980\text{—}10 \\ \hline 9.0706 \end{array}$$

121. Multiplication by Use of Logarithms

The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers. Thus, log (2 × 6) = log 2 + log 6; and log (12 × 8) = log 12 + log 8.

Example 1: Multiply 68.2 by 40.8 by using logarithms.

log (68.2 × 40.8) = log 68.2 + log 40.8.

$$\begin{aligned} \log 68.2 &= 1.8338 \\ \log 40.8 &= 1.6107 \\ \hline \log (68.2 \times 40.8) &= 3.4445 \\ \text{antilog } .4440 &= 278 \\ \text{antilog } .4445 &= 278 + x \\ \text{antilog } .4455 &= 279 \\ \frac{5}{15} &= \frac{x}{1} \\ 15x &= 5 \\ x &= .33 \\ \text{antilog } .4445 &= 2783 \\ 68.2 \times 40.8 &= 2{,}783 \end{aligned}$$

Example 2: Find the product of 2.11 and 41.3 by using logarithms.

log (2.11 × 41.3) = log 2.11 + log 41.3.

$$\begin{aligned} \log 2.11 &= 0.3243 \\ \log 41.3 &= 1.6160 \\ \hline \log (2.11 \times 41.3) &= 1.9403 \\ \text{antilog } .9400 &= 871 \\ \text{antilog } .9403 &= 871 + x \\ \text{antilog } .9405 &= 872 \\ \frac{3}{5} &= \frac{x}{1} \\ 5x &= 3 \\ x &= .6 \\ \text{antilog } 1.9403 &= 87.16 \\ 2.11 \times 41.3 &= 87.16 \end{aligned}$$

122. Division by Use of Logarithms

The logarithm of the quotient of two numbers is equal to the difference between the logarithms of the numbers. Thus, log (75 ÷ 83) = log 75 — log 83, and log (8 ÷ 2) = log 8 — log 2.

Example 1: Divide 785 by 329 by using logarithms.

log (785 ÷ 329) = log 785 — log 329.

$$\begin{aligned} \log 785 &= 2.8949 \\ \log 329 &= 2.5172 \\ \hline \log (785 \div 329) &= 0.3777 \\ \text{antilog } .3766 &= 238 \\ \text{antilog } .3777 &= 238 + x \\ \text{antilog } .3784 &= 239 \\ \frac{11}{18} &= \frac{x}{1} \\ 18x &= 11 \\ x &= .611 \\ \text{antilog } 0.3777 &= 2.386 \\ 785 \div 329 &= 2.386 \end{aligned}$$

Example 2: Find the value of $\frac{3}{7}$ by using logarithms.

$\log \frac{3}{7} = \log 3 - \log 7.$

log 3 = 0.4771
log 7 = 0.8451

Since the logarithm of 7 is greater than the logarithm of 3, it is necessary to add 10. ______ —10 to the logarithm of 3 before subtracting the logarithm of 7.

$$\begin{aligned}
\log 3 &= 10.4771-10 \\
\log 7 &= 0.8451 \\
\hline
\log (3 \div 7) &= 9.6320-10 \\
\text{antilog } .6314 &= 428 \\
\text{antilog } .6320 &= 428 + x \\
\text{antilog } .6325 &= 429 \\
\frac{6}{11} &= \frac{x}{1} \\
11x &= 6 \\
x &= .55 \\
\text{antilog } 9.6320-10 &= .42855 \\
3 \div 7 &= .42855
\end{aligned}$$

123. Finding the Power of a Number by Logarithms

The logarithm of a number raised to a power is equal to the logarithm of the number multiplied by the power.

Example 1: Evaluate $(18.7)^3$.

$$\begin{aligned}
\log (18.7)^3 &= 3 \log 18.7 \\
&= 3 \times 1.2718 \\
&= 3.8154 \\
\text{antilog } .8149 &= 653 \\
\text{antilog } .8154 &= 653 + x \\
\text{antilog } .8156 &= 654 \\
\frac{5}{7} &= \frac{x}{1} \\
7x &= 5 \\
x &= .7 \\
\text{antilog } 3.8154 &= 6537 \\
(18.7)^3 &= 6{,}537
\end{aligned}$$

Example 2: Evaluate $(.03625)^4$.

$$\begin{aligned}
\log (.03625)^4 &= 4 \log .03625 \\
\log 3620 &= .5587 \\
\log 3625 &= .5587 + x \\
\log 3630 &= .5599 \\
\frac{5}{10} &= \frac{x}{.0012} \\
x &= .0006 \\
\log (.03625)^4 &= 4\,(8.5593-10) \\
&= 34.2372-40 \\
\text{(Subtract)} \quad & \underline{30.0000-30} \\
&= 4.2372-10 \\
\text{antilog } .2355 &= 172 \\
\text{antilog } .2372 &= 172 + x \\
\text{antilog } .2380 &= 173 \\
\frac{17}{25} &= \frac{x}{1} \\
25x &= 17 \\
x &= .68 = .7 \\
\text{antilog } 4.2372-10 &= .000001727 \\
(.03625)^4 &= .000001727
\end{aligned}$$

Example 3: Evaluate $(2.13)^{\frac{2}{3}}$.

$$\begin{aligned}
\log (2.13)^{\frac{2}{3}} &= \tfrac{2}{3} \log 2.13 \\
&= \tfrac{2}{3} \times 0.3284 \\
&= 0.2189 \\
\text{antilog } .2175 &= 165 \\
\text{antilog } .2189 &= 165 + x \\
\text{antilog } .2201 &= 166 \\
\frac{14}{26} &= \frac{x}{1} \\
26x &= 14 \\
x &= .5 \\
\text{antilog } 0.2189 &= 1.655 \\
(2.13)^{\frac{2}{3}} &= 1.655
\end{aligned}$$

124. Finding the Root of a Number by Logarithms

The logarithm of the root of a number is equal to the logarithm of the number divided by the root.

Example 1: Evaluate $\sqrt[4]{34987}$.

$$\begin{aligned}
\log \sqrt[4]{34987} &= \frac{\log 34987}{4} \\
\log 34900 &= .5428 \\
\log 34987 &= .5428 + x \\
\log 35000 &= .5441 \\
\frac{87}{100} &= \frac{x}{.0013} \\
100x &= .1131 \\
x &= .0011 \\
&= \frac{4.5439}{4} \\
&= 1.135975 = 1.1360 \\
\text{antilog } .1335 &= 136 \\
\text{antilog } .1360 &= 136 + x \\
\text{antilog } .1367 &= 137 \\
\frac{25}{32} &= \frac{x}{1} \\
32x &= 25 \\
x &= .78 \\
\text{antilog } 1.1360 &= 13.678 \\
\sqrt[4]{34987} &= 13.678
\end{aligned}$$

Example 2: Evaluate $\sqrt[3]{76.24}$.

$$\begin{aligned}
\log \sqrt[3]{76.24} &= \frac{\log 76.24}{3} \\
\log 7620 &= .8820 \\
\log 7624 &= .8820 + x. \\
\log 7630 &= .8825 \\
\frac{4}{10} &= \frac{x}{.0005}
\end{aligned}$$

$$\begin{aligned} 10x &= .0020 \\ x &= .0002 \\ &= \frac{1.8822}{3} \\ &= 0.6274 \\ \text{antilog } 0.6274 &= 4.24 \\ \sqrt[3]{76.24} &= 4.24 \end{aligned}$$

Example 3: Evaluate $\sqrt[3]{.0073573}$.

$$\begin{aligned} \log \sqrt[3]{.0073573} &= \frac{\log .0073573}{3} \\ \log 73500 &= .8663 \\ \log 73573 &= .8663 + x \\ \log 73600 &= .8669 \\ \frac{73}{100} &= \frac{x}{.0006} \\ 100x &= .0438 \\ x &= .0004 \\ &= \frac{7.8667-10}{3} \end{aligned}$$

The quotient of 7.8667—10 divided by 3 is 2.6222—3⅓. By adding 20.0000—20 to 7.8667—10, the sum, 27.8667—30, can be divided by 3 and the quotient will be a workable logarithm.

$$\begin{aligned} \log .0073573 = \;& 7.8667-10 \\ \text{add } \;& \underline{20.0000-20} \\ & 27.8667-30 \end{aligned}$$

$$\begin{aligned} \frac{27.8667-30}{3} &= 9.2889-10 \\ \text{antilog } .2878 &= 194 \\ \text{antilog } .2889 &= 194 + x \\ \text{antilog } .2900 &= 195 \\ \frac{11}{22} &= \frac{x}{1} \\ 22x &= 11 \\ x &= .5 \\ \text{antilog } 9.2889-10 &= .1945 \\ \sqrt[3]{.0073573} &= .1945 \end{aligned}$$

125. Cologarithms

The *cologarithms* of a number is the logarithm of the reciprocal of the number. For example, $\text{colog } N = \log \frac{1}{N}$. However,

$$\begin{aligned} \log \frac{1}{N} &= \log 1 - \log N \\ &= 0 - \log N \\ \log \frac{1}{N} &= -\log N \end{aligned}$$

Therefore, $\text{colog } N = \log \frac{1}{N} = -\log N$. Thus the cologarithm of a number is the logarithm of the number subtracted from the logarithm of 1 (0.0000 or, to avoid a negative mantissa, 10.0000—10).

Example 1: Evaluate the cologarithm of 373.

$$\begin{aligned} \text{colog } 373 &= \log \frac{1}{373} \\ \log 1 &= 10.0000-10 \\ \log 373 &= \underline{2.5717 \qquad} \\ \text{colog } 373 &= 7.4283-10 \end{aligned}$$

Example 2: Evaluate $\frac{2.37}{3.61}$.

$$\begin{aligned} \log \frac{2.37}{3.61} &= \log 2.37 - \log 3.61 \\ &= \log 2.37 + \text{colog } 3.61 \\ \log 1 &= 10.0000-10 \\ \log 3.61 &= \underline{0.5575 \qquad} \\ \text{colog } 3.61 &= 9.4425-10 \\ \log 2.37 &= \underline{0.3747 \qquad} \\ & 9.8172-10 \\ \text{antilog } 9.8172-10 &= .65643 \end{aligned}$$

126. Computation by Logarithms

In performing logarithmic computations, follow the principles given in paragraphs 117 through 125. When negative quantities are involved (in multiplication and division), disregard the minus sign when making logarithmic calculations. After calculating the antilog, the sign is determined in accordance with the algebraic law of signs for multiplication and division.

Example 1: Evaluate $\sqrt[3]{\dfrac{(94.7)^2\,(.00789)}{(3.71)^3\,(.345)}}$.

$$\begin{aligned}
\log\ (94.7)^2 &= 2 \log 94.7 \\
&= 2 \times 1.9763 \\
&= 3.9526 \\
\log\ (.00789) &= \underline{\ 7.8971-10} \\
\log\ (94.7)^2 + \log\ (.00789) &= 11.8497-10 = 1.8497 \\
\log\ (3.71)^3 &= 3 \log 3.71 \\
&= 3 \times 0.5694 \\
&= 1.7082 \\
\log\ (.345) &= \underline{\ 9.5378-10} \\
\log\ (3.71)^3 + \log\ (.345) &= 11.2460-10 = 1.2460 \\
\log\ (94.7)^2\ (.00789) &= 1.8497 \\
\log\ (3.71)^3\ (.345) &= \frac{1.2460}{0.6037} \\
\log \sqrt[3]{\frac{(94.7)^2\,(.00789)}{(3.71)^3\,(.345)}} &= \frac{0.6037}{3} \\
&= .2012 \\
\text{antilog } .2012 &= 1.5892
\end{aligned}$$

Example 2: Evaluate $\sqrt[4]{\dfrac{(6.484)^2\,\sqrt[3]{7.667}}{(12.35)^2\,\sqrt[3]{3007}}}$.

$$\begin{aligned}
\log\ (6.484)^2 &= 2 \log 6.484 \\
&= 2 \times 0.8118 \\
&= 1.6236 \\
\log \sqrt[3]{7.667} &= \frac{\log 7.667}{3} \\
&= \frac{0.8846}{3} \\
&= 0.2949 \\
\log (6.484)^2 + \log \sqrt[3]{7.667} &= 1.6236 + .2949 \\
&= 1.9185 \\
\log\ (12.35)^2 &= 2 \log 12.35 \\
&= 2 \times 1.0917 \\
&= 2.1834 \\
\log \sqrt[3]{3007} &= \frac{\log 3007}{3} \\
&= \frac{3.4782}{3} \\
&= 1.1594 \\
\log (12.35)^2 + \log \sqrt[3]{3007} &= 2.1834 + 1.1594 \\
&= 3.3428 \\
\log\ (6.484)^2 \sqrt[3]{7.667} &= 11.9185-10 \\
\log\ (12.35)^2 \sqrt[3]{3007} &= \frac{3.3428}{8.5757-10}
\end{aligned}$$

$$\log \sqrt[4]{\frac{(6.484)^2 \sqrt[3]{7.667}}{(12.35)^2 \sqrt[3]{3007}}} = \frac{38.5757-40}{4}$$

$$= 9.6439-10$$

$$\text{antilog } 9.6439-10 = .4405$$

127. Review Problems—Logarithms

a. Find the logarithms of the following numbers to the base 10:

(1) 785

(2) 3.57

(3) .0345

(4) .000476

(5) 49.6

(6) 273.5

(7) 760.1

(8) 7.234

(9) .009875

(10) .00005254

b. Find the antilogs of the following logarithms:

(1) 4.8457

(2) 2.4330

(3) 9.5453—10

(4) 6.8299—10

(5) 0.6010

(6) 2.5690

(7) 5.4343—10

(8) 5.6994

(9) 0.2018

(10) 4.5372—10

c. Using logarithms, find the products of the following to four significant figures:

(1) 6.93×23.7

(2) 186×215

(3) 64.3×21.4

(4) $.089 \times .076$

(5) 135×42.3

d. Using logarithms, find the quotients of the following to four significant figures:

(1) $148 \div 297$

(2) $\frac{251}{648}$

(3) $14.9 \div 37.4$

(4) $47.38 \div 63.29$

(5) $\frac{1.06}{4.35}$

e. Using logarithms, evaluate the following:

(1) $(.0293)^4$

(2) $(1.756)^7$

(3) $(7.953)^8$

(4) $(69.37)^{.7}$

(5) $(27.98)^2$

(6) $\sqrt[3]{.01325}$

(7) $\sqrt[4]{815}$

(8) $\sqrt{7698}$

(9) $\sqrt[5]{8.942}$

(10) $\sqrt[4]{.000079911}$

f. Using logarithms, compute the following:

(1) $\frac{3.8 \times 2.6}{4.3}$

(2) $\sqrt[3]{\frac{.541 \times 47.3}{.0157}}$

(3) $\frac{44.1 \times 1.82}{10.27 \times .32}$

(4) $\frac{85.21 \times \sqrt[3]{4651}}{\sqrt{46.82} \times 6.230}$

(5) $\left(\frac{31.21}{40.70}\right)^3$

(6) $\sqrt[3]{\frac{(57.20)^2}{(31.42)^3}}$

(7) $\sqrt{\frac{.08152 \times 1.953}{95.27}}$

(8) $\sqrt{\frac{.8531}{9.327}} \times \sqrt[3]{\frac{518.2}{61.52}}$

(9) $\frac{48.19 \times \sqrt{56.02}}{431.6 \times \sqrt[3]{46.25} \times \sqrt{16.34}}$

(10) $\sqrt{\frac{.008150 \times .08532}{.01234 \times \sqrt[3]{.09156}}}$

CHAPTER 9

PLANE GEOMETRY

128. Introduction

Plane geometry is that part of geometry which deals with plane figures. In electronics, as in many other fields, it is necessary to know how to deal with areas of common plane figures. This chapter presents the formulas for finding the areas of triangles, quadrilaterals (plane figures having four sides and four angles), and circles. No effort has been made to cover the entire field of geometry. Only those principles and proofs are presented that are of value in practical work.

129. Definitions

a. Lines. A line has length, but no width or thickness. What is drawn on paper and called a line has thickness and breadth because of the material used to draw it—however, this mark only *represents* the actual line.

b. Angles. An angle, such as *ABC* in *A*, figure 23, is formed by the intersection of two lines. An angle, therefore, is the measure of the difference in direction of two straight lines that meet. The lines which form the angle, *AB* or *BC*, are called the *sides* of the angle, and the point of meeting, *B*, the vertex. The symbol ∠ is used to indicate angles. Angles usually are measured in *degrees*. A complete circle or rotation consists of 360 degrees. The symbol ° is used to indicate degrees; it is written to the right and slightly above the number. For example, 30 degrees is written 30°. Each degree consists of 60 *minutes*, and each minute is further broken down into 60 *seconds*. The symbol ′ is used to indicate minutes; the symbol ″ indicates seconds. For example, 20 minutes is written 20′; 15 seconds is written 15″.

(1) When one straight line is *perpendicular* to another straight line, the angle formed is a right angle (90°) (B, fig. 23).

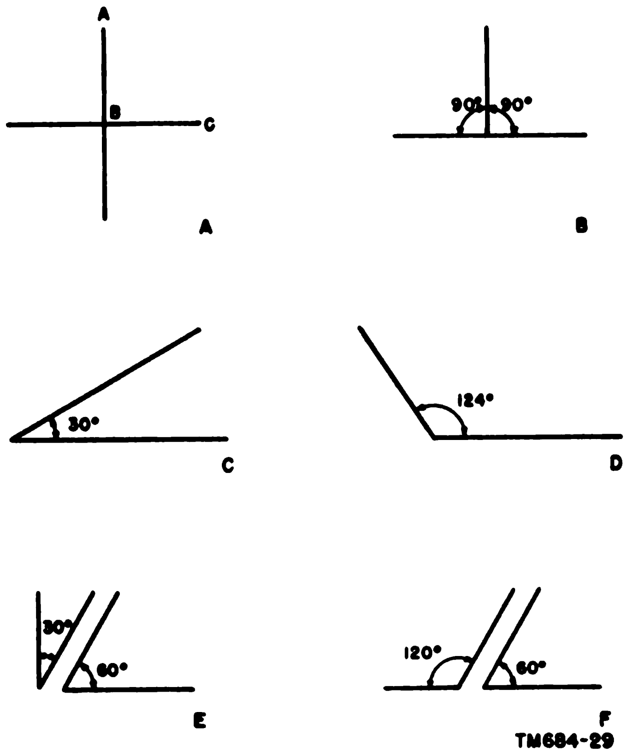

Figure 23. Angles.

(2) Two right angles, added together, form a *straight angle*. A straight angle, therefore, is an angle of 180°.

(3) Any angle less than a right angle is an *acute angle* (C, fig. 23).

(4) Any angle greater than a right angle and less than 180° is an *obtuse angle* (D, fig. 23).

(5) Two angles whose sum is one right angle are called *complementary angles* (E, fig. 23).

(6) Two angles whose sum is a straight angle are called *supplementary angles* (F, fig. 23).

130. Basic Principles of Geometric Construction

a. Reproducing Angles. To draw an angle equal to a given angle *BAC* (fig. 24)—

(1) Draw a line, *A'C'*.

(2) With *A* as the center, use a compass to strike an arc that cuts the sides of the given angle at *X* and *Y*. Using the same radius, strike a similar arc, *X'Y'*, on the line, *A'C'*.

(3) Measure the opening of the given angle by setting one point of the compass at *Y* and the other at *X*. With the compass at this distance and with *Y'* as the center, strike an arc as shown in figure 24. This will cut the first arc at point *X'*.

(4) Draw a line, *A'B'*, through *X'*. The new angle, *B'A'C'*, is the same size as angle *BAC*.

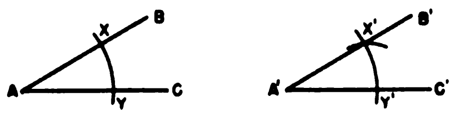

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Figure 24. Reproducing an angle.

b. Finding the Midpoint of a Straight Line Segment. To find the midpoint of any straight line segment, such as *AB* in figure 25—

(1) Use a radius greater than half the length of *AB*. Using point *A* as the center, draw arcs *CD* and *C'D'*. With point *B* as the center, and using the same radius, draw arcs *EF* and *E'F'*.

(2) Draw a straight line to connect the points where the arcs intersect. Point *X*, where this line intersects *AB*, is the midpoint of straight line segment *AB*.

c. Constructing a Perpendicular. To construct a perpendicular to a straight line at a given point—

(1) On the straight line, such as *AB* in figure 26, mark point *P* at which the perpendicular is to be constructed.

(2) Set a compass for a radius less than the shorter of the two segments, *AP* or *PB*. With *P* as a center, draw arcs, cutting line *AB* at points *X* and *Y*.

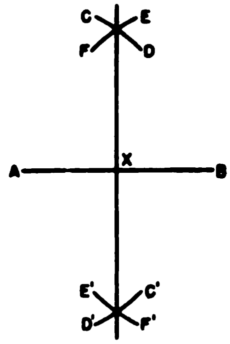

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Figure 25. Bisecting a straight line segment.

(3) Set the compass for a radius greater than *PX*. With *X* as a center, draw an arc above point *P* (fig. 26). Keep the compass at the same setting and, with *Y* as a center, draw another arc intersecting the one drawn with *X* as a center. (The two arcs may be drawn to intersect below point *P* instead of above.)

(4) Draw a straight line from the point where the two arcs intersect to point *P*. The line is perpendicular to *AB*.

(5) To construct the perpendicular bisector of a straight line segment, first find the midpoint of the line segment (*b* above), and construct the perpendicular at that point.

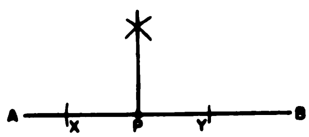

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Figure 26. Constructing a perpendicular to a straight line at a point on the line.

d. Constructing a Perpendicular to a Straight Line from a Point Not on the Line. To draw a perpendicular to a straight line from a point outside the line, such as point *P* in figure 27—

(1) With point *P* as the center, draw an arc cutting line *AB* at points *X* and *Y*.

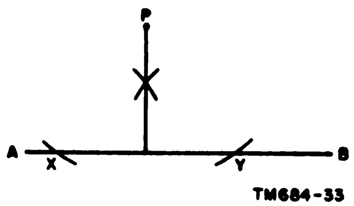

Figure 27. Constructing a perpendicular to a straight line from a point not on the line.

(2) Using a radius greater than one-half the distance between *X* and *Y* and, with points *X* and *Y* as centers, draw arcs that intersect.

(3) Draw a straight line from point *P*, through the point where the two arcs intersect, to line *AB*. The line is perpendicular to *AB*.

e. Finding the Center of a Circle.

(1) Draw any two chords, such as *AB* and *AC* in figure 28.

(2) Construct the perpendicular bisector of each chord (*c* above). Point *X*, where the two perpendicular bisectors meet, is the center of the circle.

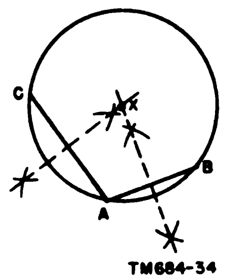

Figure 28. Finding the center of a circle.

f. Bisecting an Angle. Any angle, such as angle *CAB* in figure 29, can be divided into two equal angles. An angle, thus divided, is said to be bisected. To bisect an angle—

(1) Using *A* as a center, draw an arc cutting the sides of angle *CAB* at *X* and *Y*.

(2) With *X* and *Y* as centers, draw intersecting arcs.

(3) Draw a straight line from *A* through the point where the arcs intersect. The line divides angles *CAB* into two

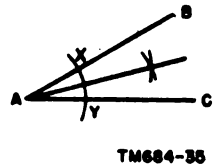

Figure 29. Bisecting an angle.

equal angles and is called the bisector of angle *CAB*.

131. Triangles

a. General. A triangle is a plane figure bounded by three straight lines. There are several different kinds of triangles.

(1) An *equilateral triangle* (A, fig. 30) has three equal sides and three equal angles; each angle equals 60°.

(2) An *isosceles triangle* has two equal

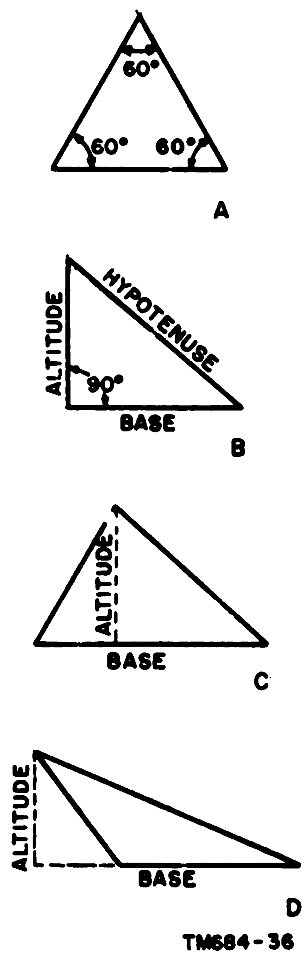

Figure 30. Triangles.

sides and two equal angles. The equal angles are opposite the equal sides.

(3) A *right triangle* (B, fig. 30) has one right angle.

(4) An *oblique triangle* (C and D, fig. 30) is one that does not contain a right angle. Thus, all except right triangles are oblique triangles.

b. Base. The base of a triangle is the side on which the triangle is supposed to stand. However, any side of a triangle may be used as the base.

c. Altitude. The altitude is the perpendicular line distance from the vertex of the triangle to the base or the base extended. In B, figure 30, the altitude of a right triangle is shown, in C, figure 30, the altitude of an acute triangle, and in D, figure 30, the altitude of an obtuse triangle. Note that in an obtuse triangle, it is necessary to extend the base of the triangle to find the altitude.

d. Area. The area of a triangle is the entire surface within the perimeter.

e. Hypotenuse. The side opposite the right angle of any right triangle is the hypotenuse (B, fig. 30).

132. Law of Angles of Any Triangle

The sum of the angles of any triangle is equal to 180°. When given any two of three angles of a triangle, the third angle can be found by subtracting the sum of the given angles from 180°

Example 1:

If two angles of a triangle are 90° and 45°, what is the size of the third angle?

$90° + 45° = 135°$

$180° - 135° = 45°$

Therefore, the third angle is 45°.

Example 2:

Angle A of triangle ABC is 100°; angle B is 30°. What is the size of angle C?

$$\angle A + \angle B + \angle C = 180°$$
$$\angle A = 100°$$
$$\angle B = 30°$$
$$\angle A + \angle B = 130°$$
$$\angle C = 180° - 130°$$
$$\angle C = 50°$$

133. Law of Right Triangles

a. The Pythagorean Theorem. This theorem, which applies to any right triangle, states that *the square of the hypotenuse is equal to the sum of the squares of the other two sides.* The Pythagorean theorem is of prime importance in trigonometry (ch. 10) since the value of one side of a right triangle can be found if the other two sides are known. Thus, in figure 31:

$$c^2 = a^2 + b^2 \text{ or } 25 = 16 + 9$$
$$a^2 = c^2 - b^2 \text{ or } 16 = 25 - 9$$
$$b^2 = c^2 - a^2 \text{ or } 9 = 25 - 16$$

Example 1: Find the hypotenuse of a right triangle if the sides are 3 and 4 inches long, respectively.

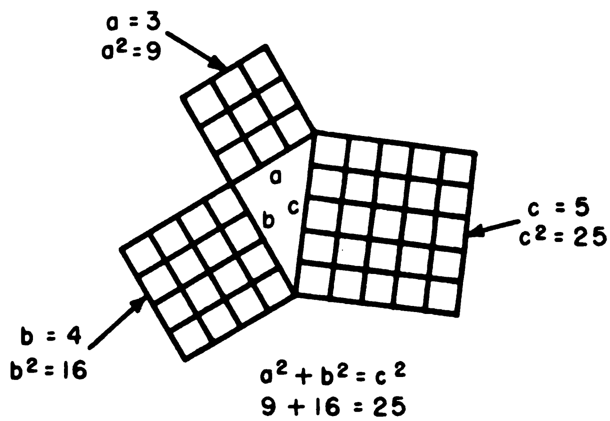

Figure 31. The Pythagorean theorem.

$c^2 = a^2 + b^2$
$c^2 = 9 + 16$
$c^2 = 25$
$c = \sqrt{25}$
$c = 5$ inches

Example 2: The hypotenuse of a right triangle is 13 inches long and one side is 5 inches long. Find the length of the other side.

$c^2 = a^2 + b^2$
$13^2 = 5^2 + b^2$
$b^2 = 169 - 25$
$b^2 = 144$
$b = \sqrt{144}$
$b = 12$ inches

Example 3: Given the right triangle *ABC* (fig. 31), find *c* if $a = 7$ and $b = 6$.

$c^2 = a^2 + b^2$
$c^2 = 49 + 36$
$c^2 = 85$
$c = \sqrt{85}$
$c = 9.22-$

	9. 2 2
	$\sqrt{85.00\ 00}$
	81
182	400
	364
1842	3600
	3684

Example 4: Given the right triangle *ABC* (fig. 31), find *b* if $a = 9$ and $c = 12$.

$b^2 = c^2 - a^2$
$b^2 = 144 - 81$
$b^2 = 63$
$b = \sqrt{63}$
$b = 7.93+$

	7. 9 3
	$\sqrt{63.00\ 00}$
	49
149	1400
	1341
1583	5900
	4749

Example 5: Given the right triangle *ABC* (fig. 31), find *a* if $b = 6$ and $c = 13$.

$a^2 = c^2 - b^2$
$a^2 = 169 - 36$
$a^2 = 133$
$a = \sqrt{133}$
$a = 11.53+$

	1 1. 5 3
	$\sqrt{01\ 33.00\ 00}$
	1
21	33
	21
225	1200
	1125
2303	7500
	6909

b. Special Right Triangles. The two right triangles in examples 1 and 2 of *a* above are special right triangles with sides that have whole numbers. These triangles are called the 3–4–5 right triangle and the 5–12–13 right triangle, although their sides may also be multiples of these numbers. For example, a triangle having sides of 6, 8, and 10 inches is also a 3–4–5 right triangle, because its sides are multiples of 3, 4, and 5. When determining the unknown side of a right triangle, the process is greatly simplified if the triangle is a 3–4–5 or 5–12–13 right triangle. In these cases, the unknown side can often be determined by inspection.

Example 1: The hypotenuse of a right triangle is 15 inches long, and one side is 12 inches long. Find the other side.

Since 15 and 12 can be divided by 3 to give 5 and 4, the triangle is a 3–4–5 right triangle. The third side, therefore, is equal to 3 times 3, or

9 inches. The answer can be checked by the Pythagorean theorem.

Example 2: The two sides of a triangle are 10 and 24 feet long. Find the length of the hypotenuse.

Dividing 10 and 24 by 2 gives 5 and 12, the two sides of a 5–12–13 right triangle. Therefore, the hypotenuse is 2 times 13, or 26 inches.

134. Area of Any Triangle

The area of any triangle is equal to one-half the product of its base and altitude. The formula for finding the area is $A = \frac{bh}{2}$ where b is the base of the triangle and h is the altitude.

Example 1:

What is the area of a triangle with a base of 15 inches and an altitude of 1 inches?

$$A = \frac{bh}{2}$$
$$= \frac{15 \times 10}{2}$$
$$= \frac{150}{2}$$
$$= 75 \text{ square inches}$$

Example 2:

Find the area of a right triangle if the base measures 7 feet and the hypotenuse 25 feet.

$$c^2 - b^2 = a^2$$
$$a^2 = 25^2 - 7^2 = 625 - 49$$
$$a^2 = 576$$
$$a = \sqrt{576} = 24 \text{ feet altitude}$$
$$A = \frac{bh}{2}$$
$$= \frac{7 \times 24}{2} = \frac{168}{2}$$
$$= 84 \text{ square feet}$$

135. Quadrilaterals

A quadrilateral is a plane figure bounded by four straight lines.

a. A *parallelogram* (A, fig. 32) is a quadrilateral having both pairs of opposite sides parallel.

b. A *rectangle* (B, fig. 32) is a parallelogram that has four right angles.

c. A *square* (C, fig. 32) is a rectangle, all four sides of which are equal.

d. A *trapezoid* (D, fig. 32) is a quadrilateral with two sides (called bases) parallel and unequal.

136. Area of Any Parallelogram

The area of any parallelogram is equal to the product of the base by the altitude. The formula for finding the area is $A = bh$ where b is the base and h is the height or altitude.

Example 1: Find the area of a square, each side of which is 15 inches.

$$A = bh$$
$$= 15 \times 15$$
$$= 225 \text{ square inches}$$

Example 2: What is the area of a rectangle with a base of 12 inches and an altitude of 7 inches?

$$A = bh$$
$$= 12 \times 7$$
$$= 84 \text{ square inches}$$

137. Area of Trapezoid

The area of a trapezoid is determined by multiplying one-half the sum of the bases by the altitude of the trapezoid.

$$\text{Thus, } A = \left(\frac{B + b}{2}\right)h.$$

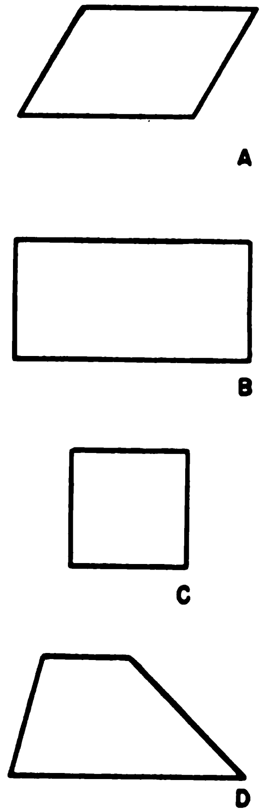

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Figure 32. Quadrilaterals.

Example: Find the area of a trapezoid the bases of which are 16 and 10 inches long and the altitude is 8 inches.

$$A = \left(\frac{B + b}{2}\right)h$$
$$= \left(\frac{16 + 10}{2}\right)8$$
$$= \frac{26}{\cancel{2}} \times \overset{4}{\cancel{8}}$$
$$= 104 \text{ square inches}$$

138. Circles

a. General. A circle is a plane figure bounded by a closed curve, every point of which is equidistant from the center.

b. Circumference. The circumference is the curved line that bounds a circle (A, fig. 33).

c. Chord. A chord is a straight line drawn through a circle and terminated at its intersections with the circumference (B, fig. 33).

d. Diameter. The diameter of a circle is a chord that passes through the center of the circle (A, fig. 33).

e. Radius. The radius of a circle is a straight line from the center to a point on the circumference (A, fig. 33). All radii of the same circle are of equal length, one-half of the diameter.

f. Arc. An arc is any part of the circumference of a circle.

g. Segment. A segment is that area of a circle bounded by a chord and the arc subtended by that chord (C, fig. 33).

h. Sector. A sector is the area between an arc and two radii drawn to the ends of the arc (C, fig. 33).

i. Tangent. A tangent is a straight line that touches the circumference of a circle at only one point and is perpendicular to the radius drawn to the point of contact (B, fig. 33). This

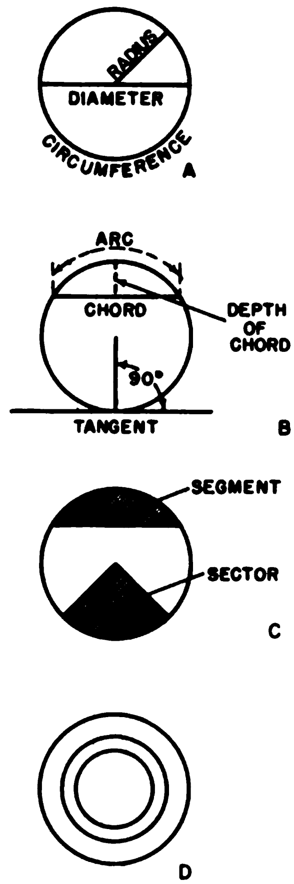

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Figure 33. Circles.

point is called the ***point of tangency*** or the ***point of contact***.

j. Concentric Circles. Concentric circles are circles having a common center (D, fig. 33).

k. Pi(π). The Greek letter π is used to represent the relationship of the circumference of any circle to its diameter. Roughly, it equals $\frac{22}{7}$. More approximately, it equals 3.1416. In many applications, it is rounded off to 3.14.

139. Circumference of Any Circle

The circumference of any circle is π times the diameter; therefore, $C = \pi D$.

Example 1: Find the circumference of a circle if the diameter is $6\frac{1}{2}$ inches.

$$C = \pi D$$
$$= 3.14 \times 6.5$$
$$= 20.42 \text{ inches}$$

Example 2: Find the diameter of a circular tank having a circumference of $31\frac{1}{2}$ inches.

When the circumference of a circle is given, the diameter is calculated by dividing the circumference by π — $D = \frac{C}{\pi}$.

$$D = \frac{C}{\pi}$$
$$= \frac{31.5}{3.1416}$$
$$= 10.03 \text{ inches}$$

140. Area of Any Circle

a. The area of any circle is equal to π multiplied by the radius squared; therefore, $a = \pi r^2$.

Example 1: Find the area of a circle having a diameter of 5 feet 6 inches.

$$A = \pi r^2$$
$$= \pi \left(\frac{5.5}{2}\right)^2$$
$$= \pi (2.75)^2$$
$$= 3.14 \times 7.56$$
$$= 23.76 \text{ square feet}$$

Example 2: What is the diameter of a circle the area of which is 78.54 square rods?

$$A = \pi r^2 \text{ and } r = \frac{D}{2}$$
$$A = \pi \left(\frac{D}{2}\right)^2$$
$$A = \frac{\pi D^2}{4}$$

Transposing:

$$D^2 = \frac{4A}{\pi}$$
$$D = \sqrt{\frac{4A}{\pi}}$$
$$D = 2\sqrt{\frac{A}{\pi}}$$

Substituting and solving for D:

$$D = 2\sqrt{\frac{78.54}{3.1416}}$$
$$D = 2\sqrt{25}$$
$$D = 2 \times 5$$
$$D = 10 \text{ rods}$$

b. The area of any circle also is equal to one-half the product of the circumference and the radius.

Example: If the diameter of a circle is 10 inches, and the circumference of the circle is 31.416 inches, what is the area of the circle?

$$A = \tfrac{1}{2}Cr$$
$$r = \tfrac{1}{2}D \text{ or } r = 5$$
$$A = \tfrac{1}{2}(31.416 \times 5)$$
$$= \frac{157.08}{2}$$
$$= 78.54 \text{ square inches}$$

141. Area of Ring

A ring is the area between the circumferences of two concentric circles. The area of a ring may be found by subtracting the area of the small circle from the area of the large circle. If R is the radius of the large circle and r is the radius of the small circle, a simplified formula for the area of the ring can be developed as follows:

$$\text{Area of ring} = \text{area of large circle} - \text{area of small circle}$$
$$= \pi R^2 - \pi r^2$$
$$= \pi (R^2 - r^2)$$

By factoring $(R^2 - r^2)$ into $(R + r)\ (R - r)$, the formula also can be written:

$$A = \pi (R + r)(R - r)$$

Example: Find the area of a ring having an inside diameter of 8 inches and an outside diameter of 12 inches.

$A = \pi(R + r)(R - r)$
$= 3.14(6 + 4)(6 - 4)$
$= 3.14 \times 10 \times 2$
$= 62.8$ square inches

142. Review Problems—Plane Geometry

a. Find the area of a rectangle having a base of 12 inches and an altitude of 8 inches.

b. What is the area of a square, each side of which is 6 inches?

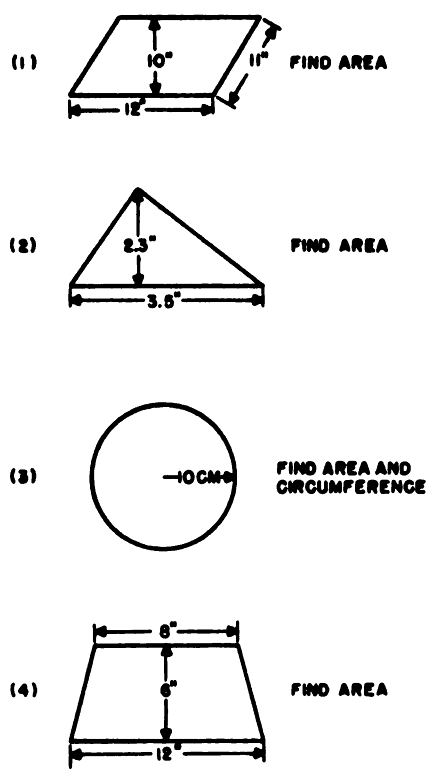
c. Find the area of a triangle of which the altitude is 5 inches and the base is 10 inches.

d. Find the area of a triangle having an altitude of 15 inches and a base of 2 inches.

e. What is the hypotenuse of a right triangle the sides of which are 12 and 8 inches?

f. Find the third side of a right triangle if one side is 7 inches and the hypotenuse is 9 inches.

g. Identify the following figures, give the formulas, and solve for the required quantity.



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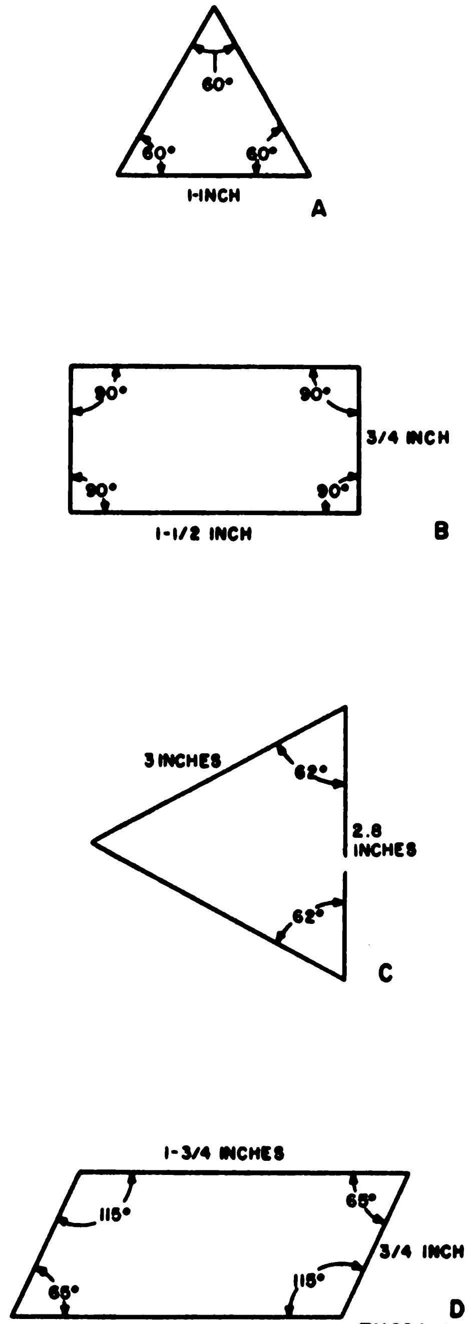
h. What are the perimeters of the following figures?



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i. Find the area of the largest circle that can be cut from a square piece of sheet metal with sides of 10 inches.

j. If the height of an antenna is 80 feet, how far from its top is an object on the ground 60 feet from the base of the pole?

k. How many square feet of lumber are needed to build 10 boxes 18 inches by 16 inches by 9 inches?

l. A metal plate is in the shape of an equilateral triangle. If the altitude is 14 inches, what is the perimeter?

CHAPTER 10

TRIGONOMETRY

Section I. BASIC TRIGONOMETRIC THEORY

143. Introduction

a. Definition. Trigonometry deals with the relationships between the sides and angles of triangles. It uses the theories of basic mathematics—the numbers of arithmetic, the equations of algebra, and the theorems of geometry—to aid in the measurement of the sides and angles of triangles.

b. Application. The ability to use angles and their trigonometric relationships in electrical calculations is especially important in the study of alternating current (ac). Most effects of ac circuit components can be studied or described only in terms of the part of a cycle by which a current lags behind a corresponding voltage, or vice versa. A large percentage of the problems relating to the analysis of ac circuits and communication networks involves the solution of the right triangle in some form. Certain facts about right triangles are familiar (ch 9)—namely, that the square of the hypotenuse is equal to the sum of the squares of the other two sides ($c^2 = a^2 + b^2$), that the sum of the acute angles of a right triangle is 90°, and that the sum of the interior angles of any triangle is 180°. However, it would be impossible to solve certain problems with only this information. After learning other relationships between the sides and angles of triangles, it will be found that trigonometry is an easy and accurate method of solving many problems in ac electricity (ch 15).

144. Trigonometric Functions

a. General. Trigonometry is based on the six trigonometric functions involved in the study of the right angle. If the value of one quantity depends on the value of a second quantity, the first quantity is said to be a function of the second. The six trigonometric functions—sine (sin), cosine (cos), tangent (tan), cotangent (cot), secant (sec), and cosecant (csc)—are derived from the ratios of the sides of a right triangle to each other.

b. The Right Triangle. Figure 34 shows a right triangle, with the angles labeled *A*, *B*, and *C*; *C* is the right angle. The sides of the triangle are labeled *a*, *b*, and *c*, with the side opposite each angle given the same letter as the angle. The following are the trigonometric ratios of the sides of a triangle:

$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\cot = \frac{\text{adjacent side}}{\text{opposite side}}$$

$$\sec = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\csc = \frac{\text{hypotenuse}}{\text{opposite side}}$$

c. Angle A. Refer again to figure 34. Using the acute angle *A*, *a* is the opposite side, *b* is the adjacent side, and *c*, which is the side opposite the right angle, is the hypotenuse. Therefore,

$$\sin A = \frac{a}{c}$$

$$\cos A = \frac{b}{c}$$

$$\tan A = \frac{a}{b}$$

$$\cot A = \frac{b}{a}$$

$$\sec A = \frac{c}{b}$$

$$\csc A = \frac{c}{a}$$

d. Angle B. Using the acute angle B in figure 34, b is the opposite side, a is the adjacent side, and c is the hypotenuse. Therefore,

$$\sin B = \frac{b}{c}$$

$$\cos B = \frac{a}{c}$$

$$\tan B = \frac{b}{a}$$

$$\cot B = \frac{a}{b}$$

$$\sec B = \frac{c}{a}$$

$$\csc B = \frac{c}{b}$$

e. Angle C. Right angle C is the angle which establishes the relationship between the other sides and other angles and thus may be called a constant. Although it is possible to obtain functions for angle C, they are not covered here because they are not needed in solving problems of this type.

Example:

Determine the values of the trigonometric functions of a right triangle with sides as follows: $a = 3$, $b = 4$, $c = 5$ (fig. 35).

Functions of angle A:

$$\sin A = \frac{a}{c} = \frac{3}{5}$$

$$\cos A = \frac{b}{c} = \frac{4}{5}$$

$$\tan A = \frac{a}{b} = \frac{3}{4}$$

$$\cot A = \frac{b}{a} = \frac{4}{3}$$

$$\sec A = \frac{c}{b} = \frac{5}{4}$$

$$\csc A = \frac{c}{a} = \frac{5}{3}$$

Functions of angle B:

$$\sin B = \frac{b}{c} = \frac{4}{5}$$

$$\cos B = \frac{a}{c} = \frac{3}{5}$$

$$\tan B = \frac{b}{a} = \frac{4}{3}$$

$$\cot B = \frac{a}{b} = \frac{3}{4}$$

$$\sec B = \frac{c}{a} = \frac{5}{3}$$

$$\csc B = \frac{c}{b} = \frac{5}{4}$$

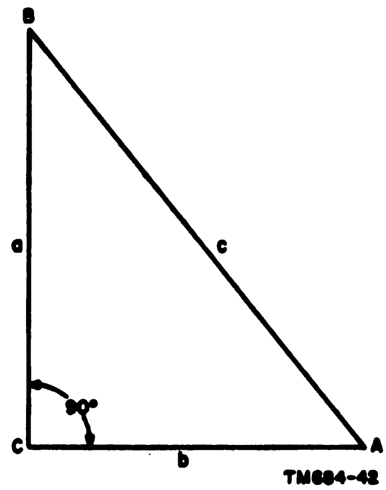


Figure 34. Trigonometric functions of the right triangle.

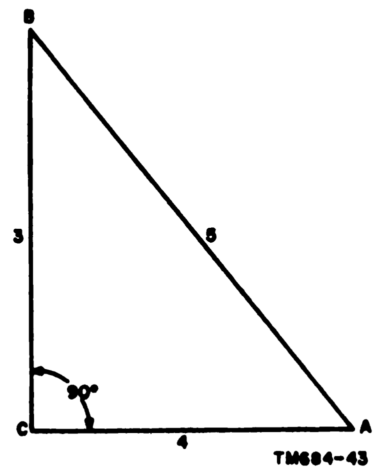


Figure 35. Right triangle with sides known.

145. Reciprocal Relations of Trigonometric Functions

From the definitions of the six trigonometric functions (par. 144), the reciprocal relations (listed below) can be determined. The cosecant, secant, and cotangent should always be thought of as the reciprocals of the sine, cosine, and tangent, respectively.

$$\sin A = \frac{a}{c} = \frac{1}{\frac{c}{a}} = \frac{1}{\csc A}$$

$$\cos A = \frac{b}{c} = \frac{1}{\frac{c}{b}} = \frac{1}{\sec A}$$

$$\tan A = \frac{a}{b} = \frac{1}{\frac{b}{a}} = \frac{1}{\cot A}$$

$$\csc A = \frac{c}{a} = \frac{1}{\frac{a}{c}} = \frac{1}{\sin A}$$

$$\sec A = \frac{c}{b} = \frac{1}{\frac{b}{c}} = \frac{1}{\cos A}$$

$$\cot A = \frac{b}{a} = \frac{1}{\frac{a}{b}} = \frac{1}{\tan A}$$

146. Functions of Complementary Angles

a. The function of an acute angle is equal to the cofunction of its complementary angle. Apply the definitions of the trigonometric functions (par. 144) to angles A and B to obtain the following relations:

$$\sin B = \frac{b}{c} = \cos A$$

$$\tan B = \frac{b}{a} = \cot A$$

$$\sec B = \frac{c}{a} = \csc A$$

$$\cos B = \frac{a}{c} = \sin A$$

$$\cot B = \frac{a}{b} = \tan A$$

$$\csc B = \frac{c}{b} = \sec A$$

b. With angle B equal to $90° - A$, these relations may be written:

$$\sin (90° - A) = \cos A$$
$$\tan (90° - A) = \cot A$$
$$\sec (90° - A) = \csc A$$
$$\cos (90° - A) = \sin A$$
$$\cot (90° - A) = \tan A$$
$$\csc (90° - A) = \sec A$$

147. Solving for Unknown Functions

If one trigonometric function of a right triangle is known, the other trigonometric functions can be determined. This is done by using the Pythagorean theorem (par. 133).

Example 1: Given the right triangle ABC (fig. 23): side a is 4; side C is 9. Since $\sin A = \frac{4}{9}$, find the other trigonometric functions of angle A.

Sin $A = \frac{a}{c}$; also, $\sin A = \frac{4}{9}$.

Therefore, $a = 4$, $c = 9$

$b^2 = c^2 - a^2$

$b^2 = 81 - 16$

$b^2 = 65$

$b = \sqrt{65}$

$b = 8.06$

```
          8. 0 6
        √65.00 00
         64
1606      10000
           9636
```

$$\sin A = \frac{4}{9} \qquad \cot A = \frac{8.06}{4}$$

$$\cos A = \frac{8.06}{9} \qquad \sec A = \frac{9}{8.06}$$

$$\tan A = \frac{4}{8.06} \qquad \csc A = \frac{9}{4}$$

Example 2: Given the right triangle *ABC* (fig. 23): side *A* is $\sqrt{3}$; side *b* is 7. Since $\tan A = \frac{\sqrt{3}}{7}$ or $\frac{1}{7}\sqrt{3}$, find the other trigonometric functions of angle *A*.

Tan $A = \frac{a}{b}$; also, $\tan A = \frac{1}{7}\sqrt{3} = \frac{\sqrt{3}}{7}$.

Therefore,

$$a = \sqrt{3},\ b = 7$$
$$c^2 = a^2 + b^2$$
$$c^2 = 3 + 49$$
$$c^2 = 52$$
$$c = \sqrt{52}$$
$$c = \sqrt{4} \cdot \sqrt{13}$$
$$c = 2\sqrt{13}$$

$$\sin A = \frac{\sqrt{3}}{2\sqrt{13}} \qquad \cot A = \frac{7}{\sqrt{3}}$$

$$\cos A = \frac{7}{2\sqrt{13}} \qquad \sec A = \frac{2\sqrt{13}}{7}$$

$$\tan A = \frac{\sqrt{3}}{7} \qquad \csc A = \frac{2\sqrt{13}}{\sqrt{3}}$$

148. Solving for Sides and Trigonometric Functions When One Side and One Function Are Given

When one side and one function of an angle of a right triangle are given, the two other sides and the remaining trigonometric functions of the given angle can be found. These are determined by use of the Pythagorean theorem.

Example 1: Given the right triangle *ABC* (fig. 34): if the hypotenuse is 30 inches and sec $A = 5$, solve for sides *a* and *b* and the trigonometric functions of angle *A*.

Sec $A = \frac{c}{b}$; also, $\sec A = \frac{30}{b}$; but $\sec A = 5$ or $\frac{5}{1}$

Therefore, $\frac{30}{b} = \frac{5}{1}$

$$5b = 30$$
$$b = 6 \text{ inches}$$

$$a^2 = c^2 - b^2$$
$$a^2 = 900 - 36$$
$$a^2 = 864$$
$$a = \sqrt{864}$$
$$a = \sqrt{144}\ \sqrt{6}$$
$$a = 12\sqrt{6} \text{ inches},\ b = 6 \text{ inches},\ c = 30 \text{ inches}$$

$$\sin A = \frac{12\sqrt{6}}{30} = \frac{12}{30}\sqrt{6} = \frac{2}{5}\sqrt{6}$$
$$\cos A = \frac{\sqrt{6}}{30} = \frac{1}{5}$$
$$\tan A = \frac{12\sqrt{6}}{6} = 2\sqrt{6}$$
$$\cot A = \frac{6}{12\sqrt{6}} = \frac{1}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{(2)(6)} = \frac{\sqrt{6}}{12} = \frac{1}{12}\sqrt{6}$$
$$\sec A = \frac{30}{6} = 5$$
$$\csc A = \frac{30}{12\sqrt{6}} = \frac{5}{2\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{(2)(6)} = \frac{5\sqrt{6}}{12} = \frac{5}{12}\sqrt{6}$$

Example 2: Given the right triangle ABC (fig. 34): solve for sides b and c and the trigonometric functions of angle A when side a is 21.2 inches and $\sin A = \frac{4}{7}$.

Sin $A = \frac{a}{c}$; also, $\sin a = \frac{21.2}{c}$, but $\sin A = \frac{4}{7}$.

Therefore, $\frac{21.2}{c} = \frac{4}{7}$
$$4c = 148.4$$
$$c = 37.1 \text{ inches}$$
$$b^2 = c^2 - a^2$$
$$b^2 = 1376.41 - 449.44$$
$$b^2 = 926.97$$
$$b = \sqrt{926.97}$$
$b = 30.4$ inches, $a = 21.2$ inches, $c = 37.1$ inches
$$\sin A = \frac{21.2}{37.1} = \frac{4}{7} \qquad \cot A = \frac{30.4}{21.2} = \frac{7.6}{5.3}$$
$$\cos A = \frac{30.4}{37.1} \qquad \sec A = \frac{37.1}{30.4}$$
$$\tan A = \frac{21.2}{30.4} = \frac{5.3}{7.6} \qquad \csc A = \frac{37.1}{21.2} = \frac{7}{4}$$

149. Constructing an Acute Angle of Right Triangle When One Trigonometric Function Is Known

When the trigonometric function of an acute angle is given, the angle may be constructed geometrically. Use the definition given for the given function.

Example: Construct the acute angle A of right triangle ABC if $\tan A = \frac{1}{4}$.

Step 1. Let $a = 1$ unit and $b = 4$ units.

Step 2. Erect perpendicular lines AC and BC. Use cross-sectional paper if available.

Step 3. Measure off 1 unit along BC and 4 units along AC (A, fig. 36).

Step 4. Join A and B, thus forming the right triangle ABC (B, fig. 36).

Step 5. Tan $A = \frac{1}{4}$; therefore, A is the required angle. Measuring angle A with a protractor shows it to

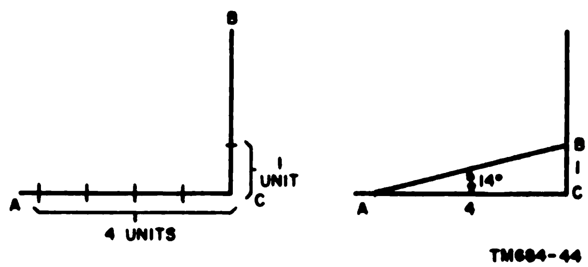


Figure 36. Constructing an angle when one function is known.

be an angle of approximately 14°.

150. Common Trigonometric Functions

a. General. There are two special-case right triangles that are commonly used in solving mathematical problems. These are the right isosceles triangle (par. 131*a*) with equal acute angles of 45° (fig. 37) and the right triangle with acute angles of 30° and 60°. The functions of these angles are tabulated in appendix III.

b. Trigonometric Functions of 45°. Draw the right triangle *ABC* (fig. 37) with angle *A* equal to 45°. Because the acute angles of a right triangle are complementary, angle *A* plus angle *B* equals 90°. Thus, angle *B* is also 45°. Since sides opposite equal angles are equal, side *a* is equal to side *b*.

Let $a = 1$ and $b = 1$.

$c^2 = a^2 + b^2$

$c^2 = 1 + 1$

$c^2 = 2$

$c = \sqrt{2}$

$\sin 45° = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}$

$\cos 45° = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1}{2}\sqrt{2}$

$\tan 45° = \frac{1}{1} = 1$

$\cot 45° = \frac{1}{1} = 1$

$\sec 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$

c. Trigonometric Functions of 30° and 60°. Draw the equilateral triangle *ABX* (fig. 38). The angles of any equilateral triangle are 60° and the sides are equal (par. 131*a*). Drop a perpendicular *BC* to the center of the base *AX*. Right angles *ACB* and *BCX* are formed by the perpendicular and the base. The angles *ABC* and *XBC* are 30° angles. Since the sides of the equilateral triangle are equal, the perpendicular bisecting the base makes the base *AC* of the right triangle *ABC* one-half the length of the base *AX* of the equilateral triangle. Thus, the side opposite the right angle in a right triangle

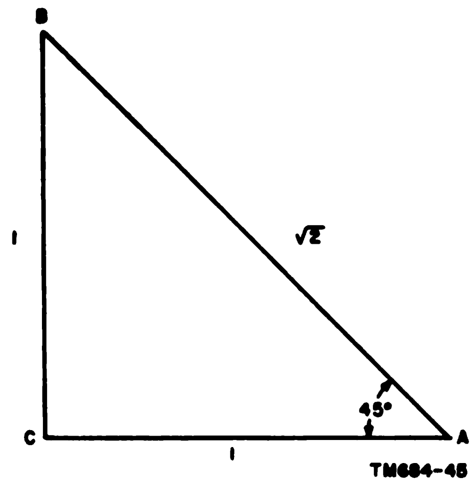


Figure 37. Right isosceles triangle—trigonometric functions of 45°.

is twice the length of the side opposite the 30° angle.

Let $b = 1$ and $c = 2$.

$a^2 = c^2 - b^2$

$a^2 = 4 - 1$

$a^2 = 3$

$a = \sqrt{3}$

$\sin 60° = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$

$\cos 60° = \frac{1}{2}$

$\tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\cot 60° = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1}{3}\sqrt{3}$

$\sec 60° = \frac{2}{1} = 2$

$\csc 60° = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2}{3}\sqrt{3}$

$\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$

$\tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1}{3}\sqrt{3}$

$\cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\sec 30° = \frac{2}{\sqrt{3}} \quad \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = \frac{2}{3}\sqrt{3}$

$\csc 30° = \frac{2}{1} = 2$

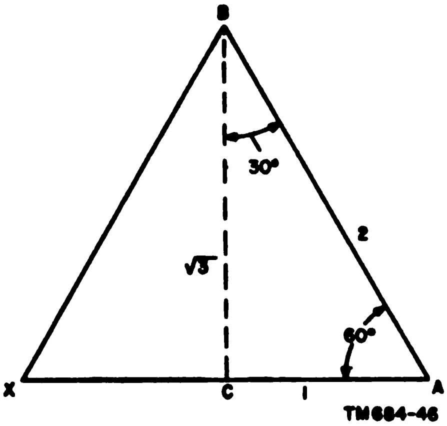


Figure 38. Equilateral right triangle—trigonometric functions of a right triangle with angles of 30° and 60°

151. Solving for Sides of 45°–45°–90° or 30°–60°–90° Triangles When One Side Is Given

In special cases, right triangles can be solved when only one side is given. These are the 45°–45°–90° isosceles triangle and the 30°–60°–90° triangle.

Example 1: Solve for the unknown sides of right triangle *ABC* if angle $A = 60°$ and $b = 4$ inches.

$\text{Tan } 60° = \frac{a}{b} = \frac{a}{4}$; however, $\tan 60° = \sqrt{3}$.

Therefore,

$$\frac{a}{4} = \frac{\sqrt{3}}{1}$$

$$a = 4\sqrt{3} \text{ inches}$$

$\text{Cos } 60° = \frac{b}{c} = \frac{4}{c}$; however, $\cos 60° = \frac{1}{2}$.

Therefore,

$$\frac{4}{c} = \frac{1}{2}$$

$$c = 8 \text{ inches}$$

Thus, $a = 4\sqrt{3}$ inches, $b = 4$ inches, $c = 8$ inches.

Example 2: Solve for the unknown sides of right triangle *ABC* if angle $A = 45°$ and $c = 6$ inches.

$\text{Sin } 45° = \frac{a}{c} = \frac{a}{6}$; however, $\sin 45° = \frac{\sqrt{2}}{2}$.

Therefore;

$$\frac{a}{6} = \frac{\sqrt{2}}{2}$$

$$2a = 6\sqrt{2}$$

$$a = 3\sqrt{2}$$

$\text{Cos } 45° = \frac{b}{c} = \frac{b}{6}$; however, $\cos 45° = \frac{\sqrt{2}}{2}$.

Therefore,

$$\frac{b}{6} = \frac{\sqrt{2}}{2}$$

$$2b = 6\sqrt{2}$$

$$b = 3\sqrt{2} \text{ inches}$$

Thus, $a = 3\sqrt{2}$ inches, $b = 3\sqrt{2}$ inches, $c = 6$ inches.

152. Calculations Involving Angles

a. Addition. To add angles, arrange the degrees, minutes, and seconds in separate columns and add each column separately. If the sum of the seconds column is 60 or more, subtract 60 or a multiple of 60 from that column, and add 1 minute or the same multiple of 1 minute to the minutes column. If the sum of the minutes column is 60 or more, subtract 60 from that column and add 1° to the degree column.

Example 1: Add 20° 40′ 25″, 8° 35′ 5″, and 30° 58′ 51″.

$$\begin{array}{rrr} 20^\circ & 40' & 25'' \\ 8^\circ & 35' & 5'' \\ 30^\circ & 58' & 51'' \\ \hline 58^\circ & 133' & 81'' \end{array}$$

Subtract 60″ from 81″ and add 1′ to 133′.

$$\begin{array}{rrr} 58^\circ & 133' & 81'' \\ & +\ 1' & -60'' \\ \hline 58^\circ & 134' & 21'' \end{array}$$

Subtract 120′ from 134′ and add 2° to 58°.

$$\begin{array}{rrr} 58^\circ & 134' & 21'' \\ +\ 2^\circ & -120' & \\ \hline 60^\circ & 14' & 21'' \end{array}$$

Example 2: Add 15° 44′ 36″ and 12° 38′ 35″.

$$\begin{array}{rrr} 15^\circ & 44' & 36'' \\ 12^\circ & 38' & 35'' \\ \hline 27^\circ & 82' & 71'' \end{array} = 27^\circ\ 83'\ 11'' = 28^\circ\ 23'\ 11''.$$

b. Subtraction. To subtract angles, arrange the degrees, minutes, and seconds in separate columns with the larger angle on top. Then, subtract the individual columns. If the upper number in a column is too small to allow subtraction, one unit must be taken away from the preceding column and 60 units added to the insufficient number to make subtraction possible.

Example 1: Subtract 14° 51′ 30″ from 86° 45′ 10″.

$$\begin{array}{rrr} 86^\circ & 45' & 10'' \\ -14^\circ & 51' & 30'' \\ \hline \end{array}$$

Subtraction cannot be performed in either the seconds or minutes columns. Subtract 1′ from 45′ leaving 44′, and add 60″ to 10″ for a total of 70″.

$$\begin{array}{rrr} 86^\circ & 44' & 70'' \\ -14^\circ & 51' & 30'' \\ \hline \end{array}$$

Subtraction still cannot be performed in the minutes column. Subtract 1° from 86°, leaving 85°, and add 60′ to 44′ for a total of 104′.

$$\begin{array}{rrr} 85^\circ & 104' & 70'' \\ -14^\circ & 51' & 30'' \\ \hline 71^\circ & 53' & 40'' \end{array}$$

Example 2: Subtract 10° 35′ 42″ from 19° 20′ 20″.

$$\begin{array}{rrr} 19^\circ & 20' & 20'' \\ -10^\circ & 35' & 42'' \\ \hline \end{array}$$

Subtraction cannot be performed in either the minutes or seconds columns. Therefore, change 19° 20′ 20″ to 18° 79′ 80″ and subtract.

$$\begin{array}{r} 18^\circ\ 79'\ 80'' \\ -10^\circ\ 35'\ 42'' \\ \hline 8^\circ\ 44'\ 38'' \end{array}$$

c. Multiplication. To multiply an angle by a given number, multiply each column by the number. If the answer in the seconds or minutes column is greater than 60, reduce as in the addition of angles (*a* above).

Example 1: Multiply 15° 21′ 40″ by 3.

$$\begin{array}{r} 15^\circ\ 21'\ \ 40'' \\ 3 \\ \hline 45^\circ\ 63'\ 120'' \end{array} = 45^\circ\ 65'\ 0'' = 46^\circ\ 5'$$

Example 2: Multiply 12° 14′ 36″ by 5.

$$\begin{array}{r} 12^\circ\ 14'\ \ 36'' \\ 5 \\ \hline 60^\circ\ 70'\ 180'' \end{array} = 60^\circ\ 73' = 61^\circ\ 13'$$

d. Division. To divide an angle by a given number, divide each column by the number (beginning with the degrees column). Change the remainder in degrees, if any, into minutes and add it to the minutes column; then, perform division on the numbers in the minutes column. Change the remainder in minutes, if any, to seconds and add it to the seconds column; then, perform division on the numbers in the seconds column.

Example 1: Divide 71° 22′ 21″ by 3.

$$\begin{array}{llll} & 23^\circ & 47' & 27'' \\ 3) & 71^\circ & 22' & 21'' \\ & 69 & & \\ & \overline{2^\circ} = & 120' & \\ & & \overline{142'} & \\ & & 141' & \\ & & \overline{1'} = & 60'' \\ & & & \overline{81''} \\ & & & 81'' \end{array}$$

Example 2: Divide 166° 17′ 36″ by 6.

$$\begin{array}{llll} & 27^\circ & 42' & 56'' \\ 6) & 166^\circ & 17' & 36'' \\ & 162^\circ & & \\ & \overline{4^\circ} = & 240' & \\ & & \overline{257'} & \\ & & 252' & \\ & & \overline{5'} = & 300'' \\ & & & \overline{336''} \\ & & & 336'' \end{array}$$

153. Review Problems—Basic Trigonometry

Note. In the following problems, angle C is the right angle and equals 90°.

a. Find the third side of each of the following right triangles ABC, if two sides are:

(1) $a = 5, b = 7$

(2) $b = 18, c = 19$

(3) $a = 17, c = 43$

(4) $a = 3b$

(5) $a = 2m, c = m^2 + 1$

b. Given the right triangle ABC, solve for the trigonometric functions of angle A in each of the following cases:

(1) $\sin A = \frac{4}{7}$

(2) $\tan A = \frac{2}{3}$

(3) $\cos A = \frac{\sqrt{3}}{2}$

(4) $\csc A = 2.4$

(5) $\cot A = \frac{1}{y}$

(6) $\sec A = 2\frac{2}{3}$

c. Solve each of the right triangles (ABC) for the two unknown sides:

(1) $\sin A = \frac{1}{2}, a = 17$

(2) $\tan A = \frac{3}{4}, b = 12$

(3) cos $A = \frac{4}{5}$, $c = 20$

(4) csc $A = \frac{15}{7}$, $c = 37.5$

(5) cot $A = \frac{3}{5}$, $a = 10$

(6) sec $A = \frac{9}{4}$, $b = 18.4$

d. Solve each of the following right triangles (*ABC*) for the unknown sides:

(1) $A = 30°$, $a = 10$

(2) $B = 45°$, $b = 7$

(3) $A = 60°$, $c = 8$

(4) $B = 30°$, $a = 9$

(5) $B = 60°$, $c = 25$

Section II. NATURAL TRIGONOMETRIC FUNCTIONS

154. Tables and Their Uses

For convenience in computing, trigonometric functions are arranged in tables similar to the tables of logarithms. The ratios themselves are called *natural* sines, cosines, tangents, cotangents, etc. The tables in appendix III give the sines and cosines, the tangents and cotangents, and the secants and cosecants of the angles from 0° to 90°. Angles less than 45° are read down the page; the degrees are at the top of the page and the minutes are on the left. Angles greater than 45° are read up the page; the degrees are at the bottom of the page and the minutes are on the right. As with logarithms, it is necessary to interpolate to find the function of an angle which does not reduce to an integral number of minutes. When working with the sine and tangent, which are increasing in size from 0° to 90°, it is necessary to add in interpolation. When working with the cosine and cotangent, which are decreasing in size from 0° to 90°, it is necessary to subtract.

155. Finding the Function of an Angle From the Table

To find the function of an angle from the table, proceed much the same as with the table of logarithms. This is illustrated by the following examples:

a. When an Angle Is Given in the Table.

Example 1: Find the cosine of 44° 27′

Step 1. Turn to the table of sines and cosines.

Step 2. Locate the 44° column at the top of the page.

Step 3. Locate the 27′ at the left of the page.

Step 4. Read .71386 in the column headed Cosin.

Step 5. Cos 44° 27′ = .71386.

Example 2: Fine the tangent of 86° 18′.

Step 1. Turn to the table of tangents and cotangents.

Step 2. Locate the 86° column at the bottom of the page.

Step 3. Locate the 18′ at the right of the page.

Step 4. Read 15.4638 in the column headed Tang.

Step 5. Tan 86° 18′ = 15.4638.

b. When an Angle Is Not Given in the Table.

Example 1: Find the sine of 32° 46′ 36″.

$$\begin{aligned} \sin 32°\ 46' &= .54122 \\ \sin 32°\ 46'\ 36'' &= .54122 + {}^{x} \\ \sin 32°\ 47' &= .54146 \end{aligned}$$

$$\begin{array}{ll} \sin 32° \; 46' \; 36'' & 32' \; 47' \\ \underline{-32° \; 46'} & \underline{-32° \; 46'} \\ 36'' & 1' = 60'' \end{array}$$

$$\text{ratio} = \frac{36}{60} = \frac{6}{10} = \frac{3}{5}$$

$$.54146 - .54122 = .00024$$

$$\text{ratio} = \frac{x}{.00024}$$

$$\frac{3}{5} = \frac{x}{.00024}$$

$$5x = .00072$$

$$x = .000144$$

$$\sin 32° \; 46' \; 36'' = .54122 + .000144 = .54136$$

Example 2: Find the tangent of 56° 43′ 27″.

$$\tan 56° \; 43' = 1.52332$$

$$\tan 56° \; 43' \; 27'' = 1.52332 + x$$

$$\tan 56° \; 44' = 1.52429$$

$$\frac{27}{60} \text{ or } \frac{9}{20} = \frac{x}{.00097}$$

$$20x = .00873$$

$$x = .000436 \text{ or } .00044$$

$$\tan 56° \; 43' \; 27'' = 1.52332 + .00044 = 1.52376$$

156. Finding an Angle When the Trigonometric Function Is Given

The procedure for using the table to find an angle corresponding to a function is similar to that of logarithms. This is illustrated in the examples in *a* and *b* below.

a. When the Function Is Given in the Table.

Example: Find the value of angle A if sine $A = .27284$.

Step 1. Find .27284 in the Sine column of the Sines and Cosines table.

Step 2. Reading 15° at the top of the column and 50′ in the minutes column on the left, angle $A = 15° \; 50'$.

b. When the Function Is Not Given in the Table.

Example 1: Find the value of angle A when sine $A = .78112$.

$$.78098 = \sin 51° \; 21'$$

$$.78112 = \sin 51° \; 21' + x$$

$$.78116 = \sin 51° \; 22'$$

$$\begin{array}{ll} .78112 & .78116 \\ \underline{-.78098} & \underline{-.78098} \\ .00014 & .00018 \end{array}$$

$$\text{ratio} = \frac{.00014}{.00018} = \frac{14}{18} = \frac{7}{9}$$

$$51° \; 22' - 51° \; 21' = 1' = 60''$$

$$\text{ratio} = \frac{x}{60}$$

$$\frac{7}{9} = \frac{x}{60}$$

$$9x = 420$$

$$x = 47$$

$$\text{angle } A = 51° \; 21' \; 47''$$

Example. Find the value of angle A when $\cot A = .33820$.

$$.33848 = \cot 71^\circ\ 18'$$
$$.33820 = \cot 71^\circ\ 18' + x$$
$$.33816 = \cot 71^\circ\ 19'$$
$$\frac{28}{32} \text{ or } \frac{7}{8} = \frac{x}{60}$$
$$8x = 420$$
$$x = 53$$
$$\text{angle } A = 71^\circ\ 18'\ 53''$$

157. Solving a Right Triangle When an Acute Angle and the Hypotenuse Are Given

To solve for the unknowns in a right triangle when an acute angle and the hypotenuse are given. proceed as in *a* and *b* below. In both examples, angle C is the right angle; therefore, angle $C = 90^\circ$.

Example 1: Find the unknown sides a and b, and the value of angle B in right triangle ABC (fig. 39) if angle A is $33^\circ\ 15'$ and the hypotenuse, c is 9 inches.

$$\angle A + \angle B + \angle C = 180^\circ$$
$$\angle B = 180^\circ - \angle A - \angle C$$
$$\angle B = 180^\circ - 33^\circ\ 15' - 90^\circ$$
$$\angle B = 56^\circ\ 45'$$
$$\sin A = \frac{a}{c}$$
$$\sin 33^\circ\ 15' = \frac{a}{9}$$
$$a = 9 \sin 33^\circ\ 15'$$
$$a = 9 \times .54829 = 4.93461$$
$$a = 4.93461$$
$$\text{Cos } A = \frac{b}{c}$$
$$\cos 33^\circ\ 15' = \frac{b}{9}$$
$$b = 9 \cos 33^\circ\ 15'$$
$$b = 9 \times .83629$$
$$b = 7.52661$$

Therefore,
$\angle A = 33^\circ\ 15'$ $\quad a = 4.93461$ inches
$\angle B = 56^\circ\ 45'$ $\quad b = 7.52661$ inches
$\angle C = 90^\circ$ $\quad c = 9$ inches

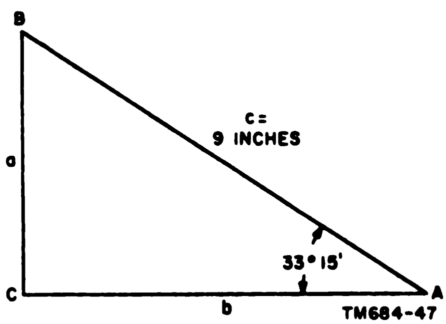


Figure 39. Solving a right triangle when an acute angle (33° 15') and the hypotenuse are given.

Example 2: **Solve for the unknown sides a and b, and the value of angle B in right triangle ABC (fig. 40) if angle A is 24° 35′ 36″ and the hypotenuse, c, is 12 inches.**

$$\angle B = 180° - \angle A - \angle C$$
$$\angle B = 180° - 24°\ 35'\ 36'' - 90°$$
$$\angle B = 65°\ 24'\ 24''$$

$$\sin A = \frac{a}{c}$$

$$\sin 24°\ 35'\ 36'' = \frac{a}{12}$$

$$a = 12 \sin 24°\ 35'\ 36''$$
$$\sin 24°\ 35' = .41602$$
$$\sin 24°\ 35'\ 36'' = .41602 + x$$
$$\sin 24°\ 36' = .41628$$
$$\frac{36}{60} \text{ or } \frac{3}{5} = \frac{x}{.00026}$$
$$5x = .00078$$
$$x = .00016$$
$$\sin 24°\ 35'\ 36'' = .41602 + .00016 = .41618$$
$$a = 12 \times .41618$$
$$a = 4.99416$$

$$\cos A = \frac{b}{c}$$

$$\cos 24°\ 35'\ 36'' = \frac{b}{12}$$

$$b = 12 \cos 24°\ 35'\ 36''$$
$$\cos 24°\ 35' = .90936$$
$$\cos 24°\ 35'\ 36'' = .90936 - x$$
$$\cos 24°\ 36' = .90924$$
$$\frac{36}{60} \text{ or } \frac{3}{5} = \frac{x}{.00012}$$
$$5x = .00036$$
$$x = .00007$$
$$\cos 24°\ 35'\ 36'' = .90936 - .00007 = .90929$$
$$b = 12 \times .90929$$
$$b = 10.91148$$

Therefore,

$\angle A = 24°\ 35'\ 36''$	$a = 4.99416$ inches
$\angle B = 65°\ 24'\ 24''$	$b = 10.91148$ inches
$\angle C = 90°$	$c = 12$ inches

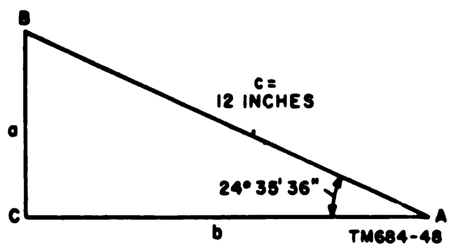


Figure 40. Solving a right triangle when an acute angle (24°35′36″) and the hypotenuse are given.

158. Solving a Right Triangle When an Acute Angle and the Adjacent Side Are Given

To solve a right triangle when an acute angle and the adjacent side are given, proceed as shown in the example below. Angle C is the right angle.

Example: Find the unknown sides a and c and the value of angle B in the right triangle ABC (fig. 41) if angle A is 37° 42′ 42″ and the side adjacent to angle A is 8 inches.

$$\angle B = 180° - 90° - 37°\ 42'\ 42''$$

$$\angle B = 52°\ 17'\ 18''$$

$$\cos A = \frac{b}{c}$$

$$\cos 37°\ 42'\ 42'' = \frac{8}{c}$$

$$c\ (\cos 37°\ 42'\ 42'') = 8$$

$$\cos 37°\ 42' = .79122$$

$$\cos 37°\ 42'\ 42'' = .79122 - x$$

$$\cos 37°\ 43' = .79105$$

$$\frac{42}{60} \text{ or } \frac{7}{10} = \frac{x}{.00017}$$

$$10x = .00119$$

$$x = .00012$$

$$\cos 37°\ 42'\ 42'' = .79122 - .00012 = .79110$$

$$.79110c = 8$$

$$c = \frac{8}{.79110}$$

$$c = 10.11$$

$$\tan A = \frac{a}{b}$$

$$\tan 37°\ 42'\ 42'' = \frac{a}{8}$$

$$a = 8 \tan 37°\ 42'\ 42''$$

$$\tan 37°\ 42' = .77289$$

$$\tan 37°\ 42'\ 42'' = .77289 + x$$

$$\tan 37°\ 43' = .77335$$

$$\frac{42}{60} \text{ or } \frac{7}{10} = \frac{x}{.00046}$$

$$10x = .00322$$

$$x = .00032$$

$$\tan 37°\ 42'\ 42'' = .77289 + .00032 = .77321$$

$$a = 8 \times .77321$$

$$a = 6.18568$$

Therefore,

$\angle A = 37°\ 42'\ 42''$ $\quad a = 6.18568$ inches

$\angle B = 52°\ 17'\ 18''$ $\quad b = 8$ inches

$\angle C = 90°$ $\quad c = 10.11$ inches

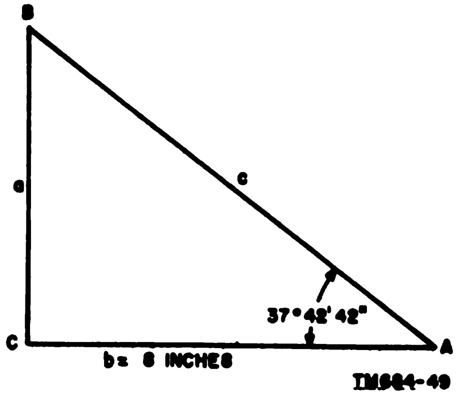


Figure 41. Solving a right triangle when an acute angle and the adjacent side are given.

159. Solving a Right Triangle When Hypotenuse and One Side Are Given

Given the hypotenuse and one other side of a right triangle, solve for the unknown angles and side as illustrated in the example below.

Example: Find the unknown angles A and B, and side c of right triangle ABC (fig. 42) if the hypotenuse is 12 inches and the side opposite angle A is 8 inches.

$$b^2 = c^2 - a^2$$
$$b^2 = 12^2 - 8^2$$
$$b^2 = 144 - 64$$
$$b^2 = 80$$
$$b = \sqrt{80}$$
$$b = 8.94$$
$$\sin A = \frac{a}{c}$$
$$\sin A = \frac{8}{12} = \frac{2}{3}$$
$$\sin A = .66667$$
$$.66653 = \sin 41°\ 48'$$
$$.66667 = \sin 41°\ 48' + x$$
$$.66675 = \sin 41°\ 49'$$
$$\frac{14}{22} = \frac{x}{60}$$
$$22x = 840$$
$$x = \frac{840}{22} = 38$$
$$.66667 = \sin 41°\ 48'\ 38''$$
$$\text{angle } A = 41°\ 48'\ 38''$$
$$\angle B = 180° - \angle C - \angle A$$
$$\angle B = 180° - 90° - 41°\ 48'\ 38''$$
$$\angle B = 48°\ 11'\ 22''$$

Therefore, $\angle A = 41°\ 48'\ 38''$ $\quad a = 8$ inches
$\angle B = 48°\ 11'\ 22''$ $\quad b = 8.94$ inches
$\angle C = 90°$ $\quad c = 12$ inches

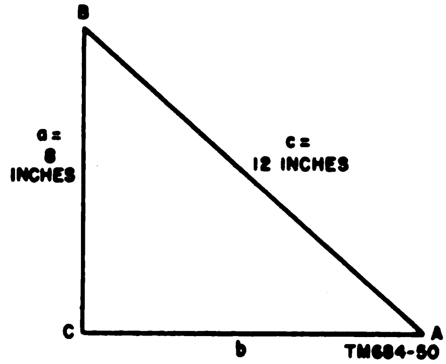


Figure 42. Solving a right triangle, when the hypotenuse and one side are given.

160. Solving a Right Triangle When Two Sides Are Given

When two sides of a right triangle are given, solve for the unknown angles and the hypotenuse as shown in the example below.

Example: Find the unknown angles A and B and side c in right triangle ABC (fig. 43) if side a is 8 inches and side b is 10 inches.

$$c^2 = a^2 + b^2$$
$$c^2 = 64 + 100$$
$$c^2 = 164$$
$$c = \sqrt{164}$$
$$c = 12.8$$
$$\tan A = \frac{a}{b}$$
$$\tan A = \frac{8}{10}$$
$$\tan A = .80000$$
$$.79972 = \tan 38° \ 39'$$
$$.80000 = \tan 38° \ 39' + x$$
$$.80020 = \tan 38° \ 40'$$
$$\frac{28}{48} \text{ or } \frac{7}{12} = \frac{x}{60}$$
$$12x = 420$$
$$x = 35$$
$$.80000 = \tan 38° \ 39' \ 35''$$
$$\text{angle } A = 38° \ 39' \ 35''$$
$$\angle B = 180° - \angle C - \angle A$$
$$\angle B = 180° - 90° - 38° \ 39' \ 35''$$
$$\angle B = 51° \ 20' \ 25''$$

Therefore,

$\angle A = 38° \ 39' \ 35''$	$a = 8$ inches
$\angle B = 51° \ 20' \ 25''$	$b = 10$ inches
$\angle C = 90°$	$c = 12.8$ inches

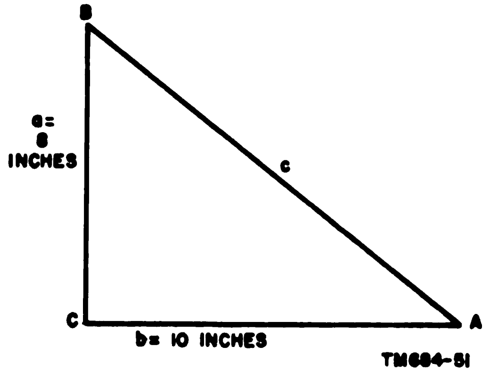


Figure 43. Solving a right triangle when two sides are given.

161. Solving a 30°–60°–90° Triangle When One Side Is Given

In a 30°–60°–90° triangle, the side opposite the 30° angle is equal to one-half the hypotenuse. Refer to paragraph 150*c* for the derivation of the trigonometric functions. Solve for the unknown sides as shown in the example below.

Example: Find the unknown sides b and c of 30°–60°–90° triangle *ABC* (fig. 44) if the side opposite the 60° angle is 6 inches.

$$\sin 60° = \frac{\sqrt{3}}{2}; \text{ also, } \sin 60° = \frac{a}{c} = \frac{6}{c}$$

$$\frac{\sqrt{3}}{2} = \frac{6}{c}$$

$$\sqrt{3}c = 12$$

$$c = \frac{12}{\sqrt{3}}$$

Eliminate $\sqrt{3}$ in the denominator by multiplying $\frac{12}{\sqrt{3}}$ by $\frac{\sqrt{3}}{\sqrt{3}}$:

$$c = \frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{\sqrt{9}} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}$$

$$c = 4\sqrt{3} = 4 \times 1.7321 = 6.9284$$

$$\tan 60° = \frac{\sqrt{3}}{1}; \text{ also, } \tan 60° = \frac{a}{b} = \frac{6}{b}$$

$$\frac{\sqrt{3}}{1} = \frac{6}{b}$$

$$\sqrt{3}b = 6$$

$$b = \frac{6}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{9}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

$$b = 2\sqrt{3} = 2 \times 1.7321 = 3.4642$$

Therefore, $a = 6$ inches
$b = 3.4642$ inches
$c = 6.9284$ inches

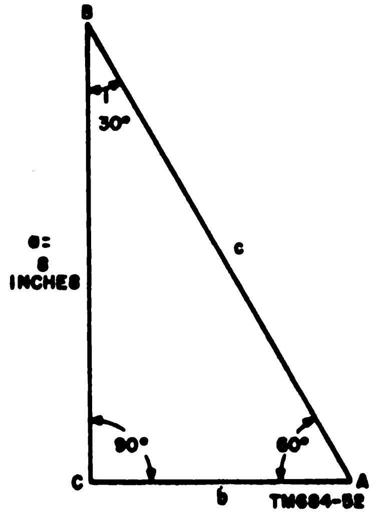


Figure 44. Solving a 30°–60°–90° triangle when one side is given.

162. Solving a 45°–45°–90° Triangle When One Side is Given

In a 45°–45°–90° triangle, the sides opposite the equal angles are equal. Refer to paragraph 150*b* for the derivation of the trigonometric functions. Solve for the unknown sides as shown in the example below.

Example: Find the unknown sides *a*, *b*, and *c* of 45°–45°–90° triangle *ABC* (fig. 45) if the side opposite acute angle *A* is 5 inches.

$$\sin 45^\circ = \frac{1}{\sqrt{2}}; \text{ also, } \sin A = \frac{a}{c} = \frac{5}{c}$$

$$\frac{1}{\sqrt{2}} = \frac{5}{c}$$

$$c = 5\sqrt{2}$$

$$c = 5 \times 1.4142 = 7.0710$$

$$\tan 45^\circ = \frac{1}{1}; \text{ also, } \tan A = \frac{a}{b} = \frac{5}{b}$$

$$\frac{1}{1} = \frac{5}{b}$$

$$[b = 5]$$

Therefore, $a = 5$ inches

$b = 5$ inches

$c = 7.071$ inches

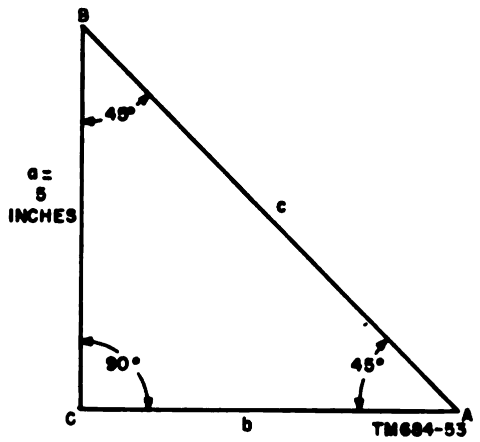


Figure 45. Solving a 45°–45°–90° triangle when one side is given.

163. Angles of Elevation and Depression

When an object is higher than the observer's eye, the angle between the horizontal and the line of sight to the object is called the *angle of elevation* (A, fig. 46). When an object is lower than the observer's eye, the angle between the line of sight to the object and the horizontal is called the *angle of depression* (B, fig. 46).

Example:

A television antenna mast is 450 feet high (fig. 47). Find to the nearest second the angle of elevation to its top at a point 200 feet from the base of the mast.

$$\tan A = \frac{a}{b}$$

$$\tan A = \frac{450}{200}$$

$$\tan A = 2.2500$$

$$2.2496 = \tan 66^\circ\ 2'$$

$$2.2500 = \tan 66^\circ\ 2' + x$$

$$2.2513 = \tan 66^\circ\ 3'$$

$$\frac{4}{17} = \frac{x}{60}$$

$$17x = 240$$

$$x = 14$$

$$2.2500 = \tan 66^\circ\ 2'\ 14''$$

$$A = 66^\circ\ 2'\ 14''$$

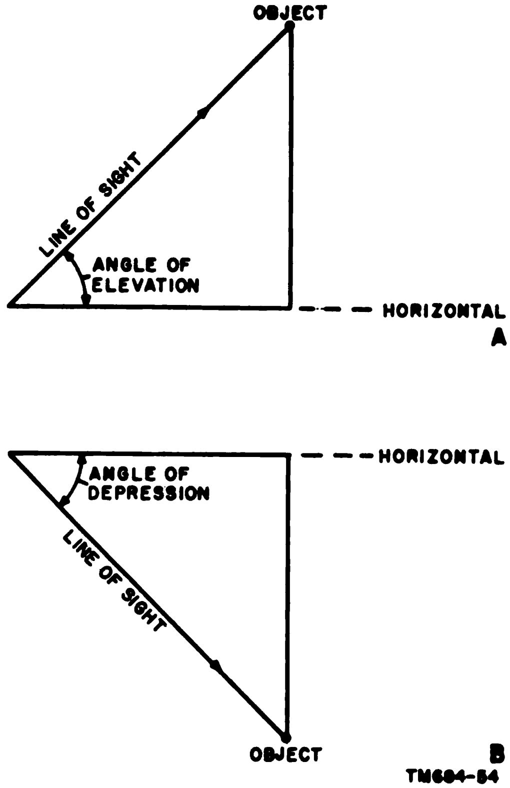


Figure 46. Angles of elevation and depression.

164. Review Problems—Natural Trigonometric Functions

a. Find the sine, cosine, tangent, and cotangent of the following angles:

(1) 1° 30′
(2) 15° 25′
(3) 32° 10′
(4) 36° 39′
(5) 44° 59′
(6) 44° 59′ 45″
(7) 35° 12′ 15″
(8) 54° 27′ 32″
(9) 48° 25′ 37″
(10) 67° 33′ 42″

b. Solve for the values of the following angles in degrees, minutes and seconds:

(1) $\sin A = .25737$
(2) $\cot A = .43279$

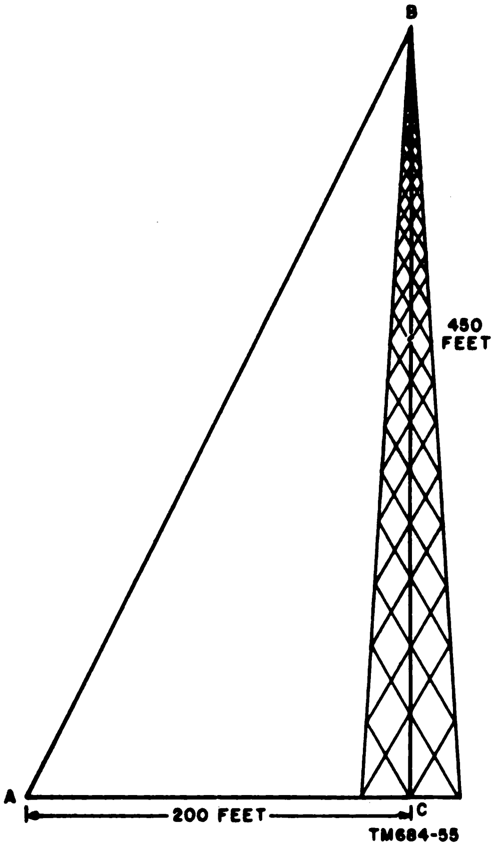


Figure 47. Finding the angle of elevation to top of an antenna mast.

(3) $\cos A = .94000$

(4) $\tan A = .47237$

(5) $\cot A = 1.17529$

(6) $\cos A = .36243$

(7) $\sin A = .37778$

(8) $\tan A = .67676$

(9) $\tan A = 1.29000$

(10) $\cot A = .79553$

c. Solve for the following (angle $C = 90°$):

(1) Angle A in right triangle ABC when $a = 19$ and $c = 27$.

(2) Side a in right triangle ABC when $A = 37° 15'$ and $c = 17$.

(3) Side c in right triangle ABC when $A = 42° 37' 15''$ and $a = 22$.

(4) Side B in right triangle ABC when $A = 37° 45' 42''$ and $c = 25$.

(5) Side c in right triangle ABC when $A = 14° 35'$ and $b = 12$.

(6) Angle A in right triangle ABC when $b = 7$ and $c = 12$.

(7) Side a in right triangle ABC when $A = 47° 22' 52''$ and $b = 31$.

(8) Side b in right triangle ABC when $A = 56° 31' 25''$ and $a = 25$.

(9) Angle A in right triangle ABC when $a = 17$ and $b = 23$.

(10) Side b in right triangle ABC when $A = 7° 32' 54''$ and $a = 17$.

(11) Side c in right triangle ABC when $a = 15$ and $b = 27$.

(12) Angle A in right triangle ABC when $a = 15$ and $b = 27$.

d. Solve the following problems:

(1) Over a distance of 300 feet, the angle of elevation of a road is 8° 24′ 30″. What is the rise in feet?

(2) The angle of elevation to the top of an antenna mast is 34° 17′ 50″. If the distance from the transit to the center of the mast is 110 feet, how high is the mast? The transit is 5 feet high.

(3) If a ladder 15 feet long just touches the top of a wall and subtends an angle of 35° 24′ 16″ with the ground, how far is the lower end of the ladder from the wall and how high is the wall?

(4) A captive balloon is anchored by 950 feet of cable. A man observes that the angle of elevation from his point of observation to the bottom of the balloon is 16° 47′ 12″. How far is he from the balloon anchor?

(5) An excavation is 33 feet wide. The angle of depression from the top of one side to the bottom of the other side is 19° 34′ 24″. How deep is the excavation?

(6) The angle of elevation from a given

point to the top of a tower is 17° 37′ 15″. Moving back 40 feet in a direct line, the angle of elevation from this point to the top of the tower is 15° 35′ 20″. Find the height of the tower.

(7) To determine the height of a tower, two sights are taken on a straight line perpendicular to the tower. If the distance between the points of observation is 60 feet and the angles of elevation are 32° 30′ 15″ and 28° 15′ 30″, respectively, what is the height of the tower?

(8) From a point in an open field a man sights on two mileposts along the side of a highway. The angles formed by an imaginary line perpendicular to the highway and the sights on the mileposts are 33° 20′ and 39° 17′ 30″. How far is the man from the closest point on the highway?

(9) An airplane is flying between two towns at an altitude of 5,000 feet. Measured with respect to the horizontal, at a given moment, the angle to the outskirts of one town is 50° 26′ 14″, while the angle to the outskirts of the other town is 64° 44′ 12″. How far apart, in a direct line, are the two towns?

(10) A radio antenna on top of a building is 10 feet high. The angle of elevation to the base of the pole is 37° 17′ 20″; the angle of elevation to the top of the antenna is 40° 30′ 15″. How high is the building?

(11) In a 45°–45°–90° right triangle the hypotenuse is 2 inches long. Find the length of the other two sides.

(12) In a 30°–60°–90° right triangle the hypotenuse is 6 inches long. Find the length of the other two sides.

Section III. TRIGONOMETRIC LAWS

165. Solving Oblique Triangles

An oblique triangle is one in which one of the angles is a right angle. The formulas in this section are used primarily to solve oblique triangles, but may also be used to solve right triangles. In the solution of triangles by trigonometric laws, the four following cases arise:

a. When any side and any two angles are given.

b. When any two sides and the angle opposite one of them are given.

c. When any two sides and the angle included between them are given.

d. When the three sides are given.

166. Law of Sines

In any triangle, the sides are proportional to the sines of the opposite angles.

Thus, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

a. Two Angles and One Side Given.

Example: Solve for the unknowns in oblique triangle *ABC* (fig. 48) when angle $A = 35° \ 47' \ 36''$, angle $B = 68° \ 42' \ 27''$, and the side opposite angle *A* is 15 inches.

$$\angle C = 180° - \angle A - \angle B$$
$$\angle C = 180° - 35° \ 47' \ 36'' - 68° \ 42' \ 27''$$
$$\angle C = 75° \ 29' \ 57''$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$b \sin A = a \sin B$$
$$b = \frac{a \sin B}{\sin A}$$
$$b = \frac{15 \sin 68° \ 42' \ 27''}{\sin 35° \ 47' \ 36''}$$

$$\sin 68° \; 42' = .93169$$
$$\sin 68° \; 42' \; 27'' = .93169 + x$$
$$\sin 68° \; 43' = .93180$$
$$\frac{27}{60} \text{ or } \frac{9}{20} = \frac{x}{.00011}$$
$$20x = .00099$$
$$x = .000049 = .00005$$
$$\sin 68° \; 42' \; 27'' = .93169 + .00005 = .93174$$
$$\sin 35° \; 47' = .58472$$
$$\sin 35° \; 47' \; 36'' = .58472 + x$$
$$\sin 35° \; 48' = .58496$$
$$\frac{36}{60} \text{ or } \frac{3}{5} = \frac{x}{.00024}$$
$$5x = .00072$$
$$x = .00014$$
$$\sin 35° \; 47' \; 36'' = .58472 + .00014 = .58486$$
$$b = \frac{15 \times .93174}{.58486}$$
$$b = \frac{13.97610}{.58486}$$
$$b = 23.89$$
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$c \sin A = a \sin C$$
$$c = \frac{a \sin C}{\sin A}$$
$$c = \frac{15 \sin 75° \; 29' \; 57''}{\sin 35° \; 47' \; 36''}$$
$$\sin 75° \; 29' = .96807$$
$$\sin 75° \; 29' \; 57'' = .96807 + x$$
$$\sin 75° \; 30' = .96815$$
$$\frac{57}{60} \text{ or } \frac{19}{20} = \frac{x}{.00008}$$
$$20x = .00152$$
$$x = .000076 = .00008$$
$$\sin 75° \; 29' \; 57'' = .96807 + .00008 = .96815$$
$$c = \frac{15 \times .96815}{.58486}$$
$$c = \frac{14.52225}{.58486}$$
$$c = 24.83$$

Therefore, $\angle A = 35° \; 47' \; 36''$ $\quad a = 15$ inches
$\angle B = 68° \; 42' \; 27''$ $\quad b = 23.89$ inches
$\angle C = 75° \; 29' \; 57''$ $\quad c = 24.83$ inches

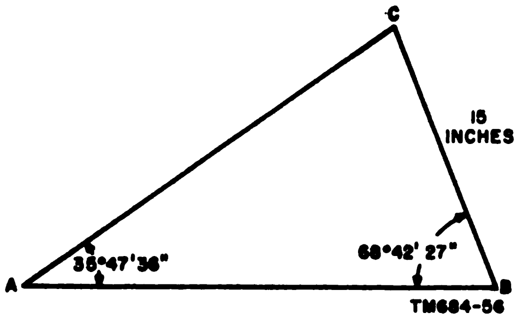


Figure 48. Solving an oblique triangle by the law of sines when two angles and a side are given.

b. Two Sides and One Angle Given.

Example: Find the unknowns in oblique triangle *ABC* (fig. 49) when angle $A = 53° \ 35' \ 40''$, the side opposite angle A is 10 inches, and the side opposite angle B is 12 inches.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a \sin B = b \sin A$$

$$\sin B = \frac{b \sin A}{a}$$

$$\sin B = \frac{12 \sin 53° \ 35' \ 40''}{10}$$

$$\sin 53° \ 35' = .80472$$

$$\sin 53° \ 35' \ 40'' = .80472 + x$$

$$\sin 53° \ 36' = .80489$$

$$\frac{40}{60} \text{ or } \frac{2}{3} = \frac{x}{.00017}$$

$$3x = .00034$$

$$x = .00011$$

$$\sin 53° \ 35' \ 40'' = .80472 + .00011 = .80483$$

$$\sin B = \frac{\overset{6}{\cancel{12}} \times .80483}{\underset{5}{\cancel{10}}}$$

$$\sin B = \frac{4.82898}{5}$$

$$\sin B = .965796 = .96580$$

$$.96578 = \sin 74° \ 58'$$

$$.96580 = \sin 74° \ 58' + x$$

$$.96585 = \sin 74° \ 59'$$

$$\frac{2}{7} = \frac{x}{60}$$

$$7x = 120$$

$$x = 17$$

$$.96580 = \sin 74° \ 58' \ 17''$$

$$\angle B = 74° \ 58' \ 17''$$

$$\angle C = 180° - \angle A - \angle B$$

$$\angle C = 180^\circ - 58^\circ\ 35'\ 40'' - 74^\circ\ 58'\ 17''$$
$$\angle C = 51^\circ\ 26'\ 3''$$
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$c \sin A = a \sin C$$
$$c = \frac{a \sin C}{\sin A}$$
$$c = \frac{10 \sin 51^\circ\ 26'\ 3''}{\sin 58^\circ\ 35'\ 40''}$$
$$\sin 51^\circ\ 26' = .78188$$
$$\sin 51^\circ\ 26'\ 3'' = .78188 + x$$
$$\sin 51^\circ\ 27' = .78206$$
$$\frac{3}{60} \text{ or } \frac{1}{20} = \frac{x}{.00018}$$
$$20x = .00018$$
$$x = .000009 = .00001$$
$$\sin 51^\circ\ 26'\ 8'' = .78188 + .00001 = .78189$$
$$c = \frac{10 \times .78189}{.80483}$$
$$c = \frac{7.8189}{.80483}$$
$$c = 9.71$$

Therefore, $\angle A = 58^\circ\ 35'\ 40''$ $\quad a = 10$ inches
$\angle B = 74^\circ\ 58'\ 17''$ $\quad b = 12$ inches
$\angle C = 51^\circ\ 26'\ 3''$ $\quad c = 9.71$ inches

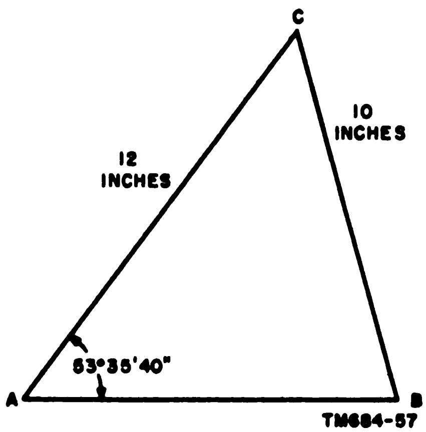


Figure 49. Solving an oblique triangle by the law of sines when two sides and an angle are given.

167. Law of Cosines

In any triangle, the square of any side equals the sum of the squares of the other two sides minus twice the product of these two sides times the cosine of the angle between them.

Thus, $a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$

Example: **Find the unknowns in oblique triangle ABC (fig. 50) when angle $C = 56° 45' 24''$, the side opposite angle A is 6 inches, and the side opposite angle B is 8 inches.**

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$c^2 = 6^2 + 8^2 - 2(6)(8) \cos 56° 45' 24''$$
$$c^2 = 36 + 64 - 96 \cos 56° 45' 24''$$
$$c^2 = 100 - 96 \cos 56° 45' 24''$$
$$\cos 56° 45' = .54829$$
$$\cos 56° 45' 24'' = .54829 - x$$
$$\cos 56° 46' = .54805$$
$$\frac{24}{60} \text{ or } \frac{2}{5} = \frac{x}{.00024}$$
$$5x = .00048$$
$$x = .000096 \text{ or } .00010$$
$$\cos 56° 45' 24'' = .54829 - .00010 = .54819$$
$$c^2 = 100 - 96(.54819)$$
$$c^2 = 100 - 52.62624$$
$$c^2 = 47.37376$$
$$c = \sqrt{47.37376}$$
$$c = 6.882$$
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$c \sin A = a \sin C$$
$$\sin A = \frac{a \sin C}{c}$$
$$\sin A = \frac{6 \sin 56° 45' 24''}{6.882}$$
$$\sin 56° 45' = .83629$$
$$\sin 56° 45' 24'' = .83629 + x$$
$$\sin 56° 46' = .83645$$
$$\frac{24}{60} \text{ or } \frac{2}{5} = \frac{x}{.00016}$$
$$5x = .00032$$
$$x = .000064 = .00006$$
$$\sin 56° 45' 24'' = .83629 + .00006 = .83635$$
$$\sin A = \frac{6(.83635)}{6.882}$$
$$\sin A = \frac{5.01810}{6.882}$$
$$\sin A = .72916$$
$$.72897 = \sin 46° 48'$$
$$.72916 = \sin 46° 48' + x$$
$$.72917 = \sin 46° 49'$$
$$\frac{19}{20} = \frac{x}{60}$$
$$20x = 1140$$
$$x = 57$$
$$.72917 = \sin 46° 48' 57''$$
$$\angle A = 46° 48' 57''$$
$$\angle B = 180° - \angle C - \angle A$$

$$\angle B = 180° - 56°\ 45'\ 24'' - 46°\ 48'\ 57''$$
$$\angle B = 76°\ 25'\ 39''$$

Therefore, $\angle A = 46°\ 48'\ 57''$ $\quad a = 6$ inches
$\angle B = 76°\ 25'\ 39''$ $\quad b = 8$ inches
$\angle C = 56°\ 45'\ 24''$ $\quad c = 6.882$ inches

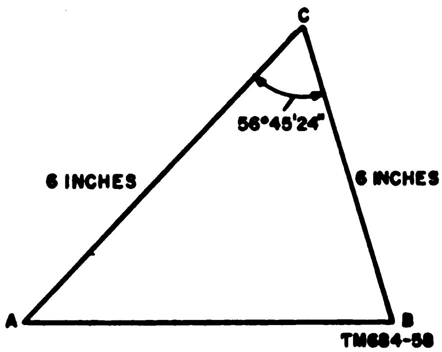


Figure 50. Solving an oblique triangle by the law of cosines when an angle and two sides are given.

168. Law of Tangents

The law of tangents is expressed by the formula $\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}$, where a and b are any two sides and A and B are the angles opposite these sides.

Example: Find the unknowns in oblique triangle ABC (fig. 51) when two sides of the triangle are 9 and 11 inches, respectively, and angle C, the angle included between these two sides, is 40° 40′ 40″.

$$\angle A + \angle B + \angle C = 180°$$
$$\angle A + \angle B + 40°\ 40'\ 40'' = 180°$$
$$\angle A + \angle B = 180° - 40°\ 40'\ 40''$$
$$\angle A + \angle B = 139°\ 19'\ 20''$$
$$\tfrac{1}{2}(A + B) = \frac{139°\ 19'\ 20''}{2}$$
$$\tfrac{1}{2}(A + B) = 69°\ 39'\ 40''$$
$$\frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)}$$
$$\frac{11 - 9}{11 + 9} \text{ or } \frac{2}{20} = \frac{\tan \frac{1}{2}(A - B)}{\tan 69°\ 39'\ 40''}$$
$$20 \tan \tfrac{1}{2}(A - B) = 2 \tan 69°\ 39'\ 40''$$
$$10 \tan \tfrac{1}{2}(A - B) = \tan 69°\ 39'\ 40''$$
$$\tan \tfrac{1}{2}(A - B) = \frac{\tan 69°\ 39'\ 40''}{10}$$
$$\tan 69°\ 39' = 2.69612$$
$$\tan 69°\ 39'\ 40'' = 2.69612 + x$$
$$\tan 69°\ 40' = 2.69853$$
$$\frac{40}{60} \text{ or } \frac{2}{3} = \frac{x}{.00241}$$
$$3x = .00482$$
$$x = .00161$$
$$\tan 69°\ 39'\ 40'' = 2.69612 + .00161 = 2.69773$$

$$\tan \tfrac{1}{2}(A - B) = \frac{2.69773}{10}$$

$$\tan \tfrac{1}{2}(A - B) = .26977$$

$$.26951 = \tan 15^\circ\ 5'$$

$$.26977 = \tan 15^\circ\ 5' + x$$

$$.26982 = \tan 15^\circ\ 6'$$

$$\frac{26}{31} = \frac{x}{60}$$

$$31x = 1560$$

$$x = 50$$

$$.26977 = \tan 15^\circ\ 5'\ 50''$$

$$\tfrac{1}{2}(A - B) = 15^\circ\ 5'\ 50''$$

$$\begin{array}{rcl} \tfrac{1}{2}(A + B) &=& \tfrac{1}{2}A + \tfrac{1}{2}B = 69^\circ\ 39'\ 40'' \\ \tfrac{1}{2}(A - B) &=& \underline{\tfrac{1}{2}A - \tfrac{1}{2}B = 15^\circ\ \ 5'\ 50''} \\ \text{(add)} && A \qquad\quad = 84^\circ\ 44'\ 90'' \end{array}$$

$$\angle A = 84^\circ\ 45'\ 30''$$

$$\begin{array}{rcl} \tfrac{1}{2}(A + B) &=& \tfrac{1}{2}A + \tfrac{1}{2}B = 69^\circ\ 38'\ 100'' \\ \tfrac{1}{2}(A - B) &=& \underline{\tfrac{1}{2}A - \tfrac{1}{2}B) = 15^\circ\ 5'\ 50''} \\ \text{(subtract)} && \qquad B = 54^\circ\ 33'\ 50'' \end{array}$$

$$\angle B = 54^\circ\ 33'\ 50''$$

$$\frac{a}{\sin A} = \frac{c}{\sin c}$$

$$c \sin A = a \sin C$$

$$c = \frac{a \sin C}{\sin A}$$

$$c = \frac{11 \sin 40^\circ\ 40'\ 40''}{\sin 84^\circ\ 45'\ 30''}$$

$$\sin 40^\circ\ 40' = .65166$$

$$\sin 40^\circ\ 40'\ 40'' = .65166 + x$$

$$\sin 40^\circ\ 40'\ 41'' = .65188$$

$$\frac{40}{60} \text{ or } \frac{2}{3} = \frac{x}{.00022}$$

$$3x = .00044$$

$$x = .000146 = .00015$$

$$\sin 40^\circ\ 40'\ 40'' = .65166 + .00015 = .65181$$

$$\sin 84^\circ\ 45' = .99580$$

$$\sin 84^\circ\ 45'\ 30'' = .99580 + x$$

$$\sin 84^\circ\ 46' = .99583$$

$$\frac{30}{60} \text{ or } \frac{1}{2} = \frac{x}{.00003}$$

$$2x = .00003$$

$$x = .000015 = .00002$$

$$\sin 84^\circ\ 45'\ 30'' = .99580 + .00002 = .99582$$

$$c = \frac{11 \sin 40^\circ\ 40'\ 40''}{\sin 84^\circ\ 45'\ 30''}$$

$$c = \frac{11 \times .65181}{.99582}$$

$$c = \frac{7.16991}{.99582}$$

$$c = 7.2$$

Therefore, $\angle A = 84° \; 45' \; 30''$ $a = 11$ inches
$\angle B = 54° \; 33' \; 50''$ $b = 9$ inches
$\angle C = 40° \; 40' \; 40''$ $c = 7.2$ inches

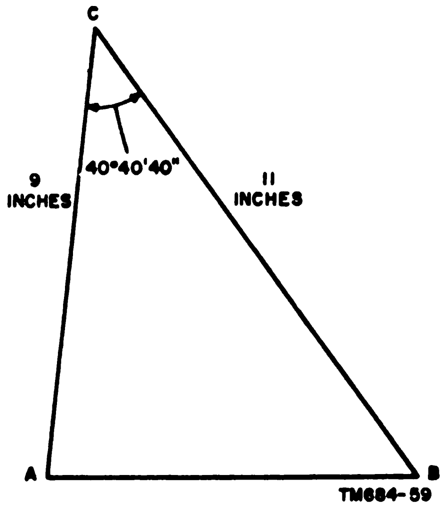


Figure 51. Solving an oblique triangle by the law of tangents when an angle and two sides are given.

169. Finding an Angle When Three Sides Are Given

The following formulas are used to find the angles of a triangle when three sides of the triangle are given:

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}$$

$$\sin \tfrac{1}{2}B = \sqrt{\frac{(s - a)(s - c)}{ac}}$$

$$\sin \tfrac{1}{2}C = \sqrt{\frac{(s - a)(s - b)}{ab}}$$

In these formulas, a, b, and c are the sides of the triangle, and $s = \tfrac{1}{2}(a + b + c)$.

Example: Find the angles of an oblique triangle if $a = 5$ inches, $b = 8$ inches, and $c = 11$ inches.

$$s = \tfrac{1}{2}(a + b + c)$$
$$s = \tfrac{1}{2}(5 + 8 + 11)$$
$$s = \tfrac{1}{2}(24)$$
$$s = 12$$
$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}$$
$$\sin \tfrac{1}{2}A = \sqrt{\frac{(12 - 8)(12 - 11)}{(8)(11)}}$$
$$\sin \tfrac{1}{2}A = \sqrt{\frac{(4)(1)}{88}}$$
$$\sin \tfrac{1}{2}A = \sqrt{\frac{4}{88}} = \sqrt{\frac{1}{22}}$$
$$\sin \tfrac{1}{2}A = \sqrt{.0454545}$$

$\sin \frac{1}{2}A = .21319$

$.21303 = \sin 12° \; 18'$

$.21319 = \sin 12° \; 18' + x$

$.21331 = \sin 12° \; 19'$

$\frac{16}{28} \text{ or } \frac{4}{7} = \frac{x}{60}$

$7x = 240$

$x = 34$

$.21319 = \sin 12° \; 18' \; 34''$

$\frac{1}{2}A = 12° \; 18' \; 34''$

$\angle A = 24° \; 36' \; 68'' \text{ or } 24° \; 37' \; 8''$

$\sin \frac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{ac}}$

$\sin \frac{1}{2}B = \sqrt{\frac{(12-5)(12-11)}{(5)(11)}}$

$\sin \frac{1}{2}B = \sqrt{\frac{(7)(1)}{55}}$

$\sin \frac{1}{2}B = \sqrt{\frac{7}{55}}$

$\sin \frac{1}{2}B = \sqrt{.1272727}$

$\sin \frac{1}{2}B = .35675$

$.35674 = \sin 20° \; 54'$

$.35675 = \sin 20° \; 54' + x$

$.35701 = \sin 20° \; 55'$

$\frac{1}{27} = \frac{x}{60}$

$27x = 60$

$x = 2$

$.35675 = \sin 20° \; 54' \; 2''$

$\frac{1}{2}B = 20° \; 54' \; 2''$

$\angle B = 40° \; 108' \; 4'' \text{ or } 41° \; 48' \; 4''$

$\angle C = 180° - \angle A - \angle B$

$\angle C = 180° - 24° \; 37' \; 8'' - 41° \; 48' \; 4''$

$\angle C = 180° - 66° \; 25' \; 12''$

$\angle C = 113° \; 34' \; 48''$

Therefore, $\angle A = 24° \; 37' \; 8''$

$\angle B = 41° \; 48' \; 4''$

$\angle C = 113° \; 34' \; 48''$

170. Finding the Area of a Triangle When Two Sides and the Included Angle Are Given

The formula for finding the area of a triangle when two sides and the included angle are given is $S = \frac{1}{2} ab \sin C$ where S is the area of the triangle, a and b are the given sides, and C is the included angle.

Example: Find the area of oblique triangle ABC (fig. 52) when two sides are 7 and 8 inches, respectively, and the included angle is 50° 50′ 50″.

$S = \frac{1}{2}ab \sin C$

$S = \frac{1}{2} \times 7 \times 8 \times \sin 50° \; 50' \; 50''$

$\sin 50° \; 50' = .77531$

$\sin 50° \; 50' \; 50'' = .77531 + x$

$$\sin 50° \; 51' = .77550$$

$$\frac{50}{60} \text{ or } \frac{5}{6} = \frac{x}{.00019}$$

$$6x = .00095$$

$$x = .00016$$

$$\sin 50° \; 50' \; 50'' = .77531 + .00016 = .77547$$

$$S = \tfrac{1}{2} \times 7 \times 8 \times .77547 = 21.71316$$

$$S = 21.71316 \text{ square inches}$$

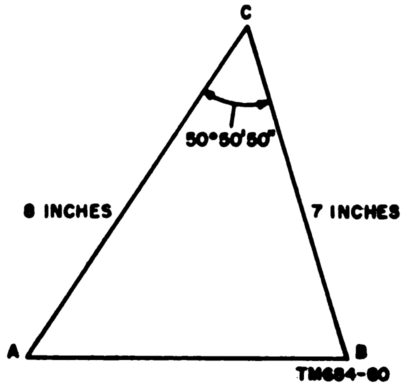


Figure 52. Solving for the area of an oblique triangle when two sides and the included angle are given.

171. Finding the Area of a Triangle When Two Angles and a Side Are Given

The formula for finding the area of a triangle when two angles and a side are given is $S = \frac{a^2 \sin B \sin C}{2 \sin A}$ where S is the area of the triangle, B and C are the given angles, and a is the given side.

Example: Find the area of oblique triangle ABC (fig. 53) when the two angles are 38° 42′ 48″ and 68° 52′ 42″ and the side is 10 inches.

$$\angle A = 180° - \angle B - \angle C$$

$$\angle A = 180° - 38° \; 42' \; 48'' - 68° \; 52' \; 42''$$

$$\angle A = 180° - 107° \; 35' \; 30''$$

$$\angle A = 72° \; 24' \; 30''$$

$$S = \frac{a^2 \sin B \sin C}{2 \sin A}$$

$$S = \frac{10^2 \sin 38° \; 42' \; 48'' \sin 68° \; 52' \; 42''}{2 \sin 72° \; 24' \; 30''}$$

$$\sin 38° \; 42' = .62524$$

$$\sin 38° \; 42' \; 48'' = .62524 + x$$

$$\sin 38° \; 43' = .62547$$

$$\frac{48}{60} \text{ or } \frac{4}{5} = \frac{x}{.00023}$$

$$5x = .00092$$

$$x = .00018$$

$$\sin 38° \; 42' \; 48'' = .62524 + .00018 = .62542$$

$$\sin 68° \; 52' = .93274$$

$$\sin 68° \; 52' \; 42'' = .93274 + x$$

$\sin 68^\circ\ 53' = .93285$

$\frac{42}{60} \text{ or } \frac{7}{10} = \frac{x}{.00011}$

$10x = .00077$

$x = .000077 \text{ or } .00008$

$\sin 68^\circ\ 53'\ 42'' = .93274 + .00008 = .93282$

$\sin 72^\circ\ 24' = .95319$

$\sin 72^\circ\ 24'\ 30'' = .95319 + x$

$\sin 72^\circ\ 25' = .95328$

$\frac{30}{60} \text{ or } \frac{1}{2} = \frac{x}{.00009}$

$2x = .00009$

$x = .000045 \text{ or } .00005$

$\sin 72^\circ\ 24'\ 30'' = .95319 + .00005 = .95324$

$S = \frac{100 \times .62542 \times .93282}{2 \times .95324}$

$S = \frac{50 \times .62542 \times .93282}{.95324}$

$S = \log 50 + \log .62542 + \log .93282 - \log .95324$

$\log 50 = 1.6990$

$\log .62500 = 9.7959-10$

$\log .62542 = 9.7959-10 + x$

$\log .62600 = 9.7966-10$

$\frac{42}{100} = \frac{x}{.0007}$

$100x = .0294$

$x = .000294 \text{ or } .0003$

$\log .62542 = 9.7959-10 + .0003 = 9.7962-10$

$\log .93200 = 9.9694-10$

$\log .93282 = 9.9694-10 + x$

$\log .93300 = 9.9699-10$

$\frac{82}{100} = \frac{x}{.0005}$

$100x = .0410$

$x = .00041 \text{ or } .0004$

$\log .93282 = 9.9694-10 + .0004 = 9.9698-10$

$\log .95300 = 9.9791-10$

$\log .95324 = 9.9791-10 + x$

$\log .95400 = 9.9795-10$

$\frac{24}{100} = \frac{x}{.0004}$

$100x = .0096$

$x = .000096 \text{ or } .0001$

$\log .95324 = 9.9791-10 + .0001 = 9.9792-10$

$S = 1.6990 + 9.7962-10 + 9.9698-10 - 9.9792-10$

$$
\begin{array}{r}
1.6990 \\
9.7962-10 \\
+\ 9.9698-10 \\
\hline
21.4650-20 \\
-\ 9.9792-10 \\
\hline
11.4858-10 \text{ or } 1.4858
\end{array}
$$

$$\text{antilog } 1.4857 = 30.6$$
$$\text{antilog } 1.4858 = 30.6 + x$$
$$\text{antilog } 1.4871 = 30.7$$
$$\frac{1}{14} = \frac{x}{.1}$$
$$14x = .1$$
$$x = .007$$
$$\text{antilog } 1.4858 = 30.6 + .007 = 30.607$$
$$S = 30.607 \text{ square inches}$$

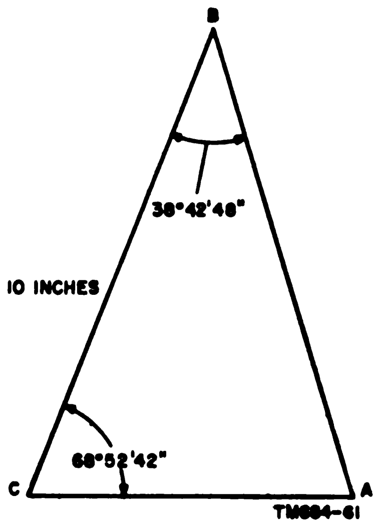


Figure 58. Solving for the area of an oblique triangle when two angles and a side are given.

172. Finding the Area of Triangle When Three Sides Are Given

To find the area of triangle when three sides are given, use the formula

$$S = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the sides of the triangle and $s = \frac{1}{2}(a + b + c)$.

Example: Find the area of an oblique triangle when the sides are 8, 11, and 15 inches, respectively.

$$s = \tfrac{1}{2}(a + b + c)$$
$$s = \tfrac{1}{2}(8 + 11 + 15)$$
$$s = \tfrac{1}{2}(34)$$
$$s = 17$$
$$S = \sqrt{s(s-a)(s-b)(s-c)}$$
$$S = \sqrt{17(17-8)(17-11)(17-15)}$$
$$S = \sqrt{17(9)(6)(2)}$$
$$S = \sqrt{1836}$$
$$S = 42.84 \text{ square inches}$$

173. Review Problems—Trigonometric Laws

a. In an oblique triangle ABC, angle $A = 42° 15' 12''$, angle $B = 75° 28' 10''$, and side b measures 21 inches. Solve the triangle for angle C and side a.

b. In an oblique triangle ABC, angle $C = 52° 30'$, side $b = 45$ inches, and side $c = 38$ inches. Solve for angle B.

c. In an oblique triangle ABC, sides a, b, and c opposite angles A, B, and C have lengths of 9, 16, and 21 inches, respectively. Find the three angles of the triangle.

d. In an oblique triangle where a and b are any two sides and A and B are the angles opposite these sides, angle $C = 57° 20' 45''$, $a = 9.73$ inches, and $b = 6.47$ inches. Find angles A and B.

e. The three sides of a triangle are 40, 37, and 13 inches, respectively. Find the area of the triangle.

f. Two sides of an oblique triangle measure 12 and 18 feet, respectively. The angle between the two sides is 115°. Find the area of the triangle.

g. In a triangle ABC, angle $A = 30°$ and angle $B = 60°$. The side opposite angle $C = 16$ inches. Find the area of the triangle.

h. In an oblique triangle ABC, angle $C = 62° 50'$. The side opposite angle A measures 9.65 inches, and the side opposite angle B measures 17.85 inches. Find angles A and B and the length of the side opposite angle C.

CHAPTER 11

RADIANS

174. Angular Measurement Using Radians

a. Definition. A radian is a unit of angular measurement equal to that angle which, when its vertex is upon the center of a circle, intercepts an arc that is equal in length to the radius of the circle. Thus, in figure 54, central angle *AOB* is equal to 1 radian because arc *AB* is equal to radius *OA*.

(1) The system that makes use of the radian is called the ***natural system*** of angular measurement because it has no arbitrary unit, such as the degree, but is founded upon the observation that the absolute size of any angle is the ratio of its arc to the radius of that arc. Where the arc and radius are equal, the ratio is 1, and this unit is the radian.

(2) The natural system of angular measurement—also called the circular system and the radian system—is used extensively in electrical formulas (part II).

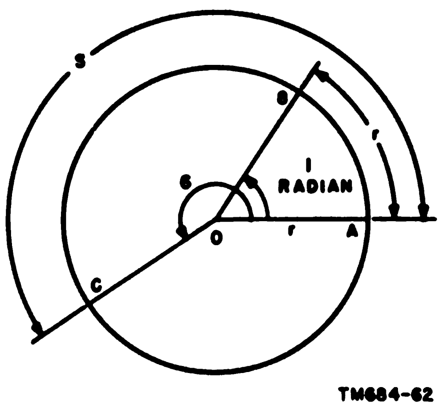


Figure 54. The radian or circular system of measurement.

b. Finding Any Angle. To find any angle, such as angle *AOC* in figure 54, when the length of arc *AB* is known, determine the number of times that radius *r* will go into arc length *ABC*, thus determining the number of radians in the angle.

Thus,

$$\text{Angle} = \frac{\text{arc}}{\text{radius}}$$

or, if angle *AOC* is denoted by the Greek letter θ (Theta) and arc *ABC* by s,

$$\theta = \frac{s}{r} \text{ radians}$$

Example: A circle has a radius of 6 inches. Find the angle subtended at the center of the circle by an arc 9 inches in length.

$$\theta = \frac{s}{r}$$
$$= \frac{9}{6}$$
$$= 1.5 \text{ radians}$$

c. Finding Length of Arc. To find the length of an arc intercepted by a central angle when the radius of the circle and the number of radians in the angle are known, use the formula in *b* above in the form—

$$s = r\theta$$

Example: A circle has a radius of 5 feet. How long is the arc intercepted by a central angle of 1.5 radians?

$$s = r\theta$$
$$= 5 \times 1.5$$
$$= 7.5 \text{ feet}$$

175. The Relation Between Degrees and Radians

a. General. It is often necessary to convert an angle from degrees to radians or from radians to degrees. If the angle is one complete revolution, the arc is one complete circumference of a circle; thus, it is 2π times the radius. Therefore, the angle is equal to $2\pi r$ divided by r—that is, 2π radians ($\pi = 3.1416$).

Therefore,

$$1 \text{ revolution} = 2\pi \text{ radians}$$

also

$$1 \text{ revolution} = 360°$$

Thus,

$$2\pi \text{ radians} = 360°$$

$$1 \text{ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = 57.29578°$$

and since

$$360° = 2\pi \text{ radians}$$

$$1° = \frac{2\pi}{360} = \frac{\pi}{180} = 0.017453 \text{ radians}$$

To change radians to degrees, accurate to seconds, use figures accurate to at least five decimal places.

b. Changing Degrees to Radians and Radians to Degrees.

Example 1: Change 2.74 radians to degrees, minutes, and seconds.

$$\begin{aligned}
1 \text{ radian} &= 57.29578° \\
2.74 \text{ radians} &= 2.74(57.29578) \\
&= 156.99044° \\
1° &= 60' \\
.99044° &= .99044(60)' \\
&= 59.4264' \\
.4264' &= .4264(60)'' \\
&= 25.5'' \\
2.74 \text{ radians} &= 156°\ 59'\ 25.5''
\end{aligned}$$

Example 2: Change 57° 15′ 18″ to radians.

Step 1. Change the minutes and seconds to decimals of a degree:

$$\begin{aligned}
1' &= 60'' \\
18'' &= \frac{18}{60} \\
&= .3' \\
15.3' &= \frac{15.3}{60} \\
&= .255° \\
57°\ 15'\ 18'' &= 57.255°
\end{aligned}$$

Step 2. Change to radians:

$$\begin{aligned}
1° &= .017453 \text{ radian} \\
57.255° &= 57.255(.017453) \\
&= .99927 \text{ radian}
\end{aligned}$$

c. Expressing Angles in Radians as Multiples of π. It is often convenient to express angles in radians as multiples of π. Since 360° = 2π radians, 90° = $\frac{1}{2}\pi$ radians, 40° = $\frac{1}{4}\pi$ radians, etc. It is necessary only to multiply the degrees by $\frac{\pi}{180}$ to change to radians.

Example: Express 135° in radians as a multiple of π.

$$135° = 135\left(\frac{\pi}{180}\right)$$
$$= \frac{3}{4}\pi \text{ radians}$$

176. Review Problems—Radians

a. Find the angle θ for the following arc lengths and radii:

(1) $r = 5$ inches, $s = 2$ inches.
(2) $r = 3$ feet, $s = 12$ feet.
(3) $r = .8$ miles, $s = 6.4$ miles.
(4) $r = 27$ meters, $s = 75$ meters

b. Find the arc lengths for the following angles and radii:

(1) $\theta = 5$ radians, $r = 7$ inches
(2) $\theta = 8$ radians, $r = 2.2$ feet
(3) $\theta = 2.1$ radians, $r = 9$ miles
(4) $\theta = .03$ radians, $r = .066$ inch

c. Express the following angles in radians:

(1) 30°
(2) 263° 12′
(3) 158° 33′
(4) 336° 24′ 22″

d. Express the following angles in degrees:

(1) [illegible] radians
(2) 25 radians
(3) 3.45 radians
(4) 3π radians

e. Express the following angles as multiples of π:

(1) 30°
(2) 60°
(3) 225°
(4) 720°

CHAPTER 12

VECTORS

177. Plane Vectors

a. A line segment used to represent a quantity that has direction as well as magnitude is called a vector. The length of a vector is proportionate to the magnitude, and the arrow, or head, of the vector indicates the direction of the quantity represented.

b. The quantity represented by a vector is called a vector quantity. This is the directed magnitude itself. Electrical quantities, such as current and voltage, are vector quantities in ac circuits (par. 194).

Example: An airplane is flying northeast at 120 miles per hour. Its speed is represented on figure 55 by line *OA*. The direction in which the airplane is traveling is represented by the direction of the line.

178. Vector Notation

Because a vector quantity has direction as well as magnitude, the methods of denoting a vector are different from the methods of denoting a scaler quantity. A vector may be denoted by two letters, the first indicating the origin, or initial point, and the other indicating the head or terminal point. For example, a vector may be represented by the letters *AB*, indicating that the quantity went from *A* to *B*. A small arrow sometimes is placed over the letters for emphasis; for example, $\overrightarrow{AB}$. Another method of notation is $A\underline{/\theta}$, where *A* represents the magnitude of the quantity, and $\underline{/\theta}$ represents the angle the vector makes with some reference line. For example, if line *OE* in figure 55 were used as the reference line, vector *OA* could be represented by the notation $120\underline{/45°}$, where 120 represents the magnitude of the quantity, and $\underline{/45°}$ represents the direction with respect to line *OE*. With respect to line *ON*, vector *OA*, would be represented by the notation $120\underline{/-45°}$.

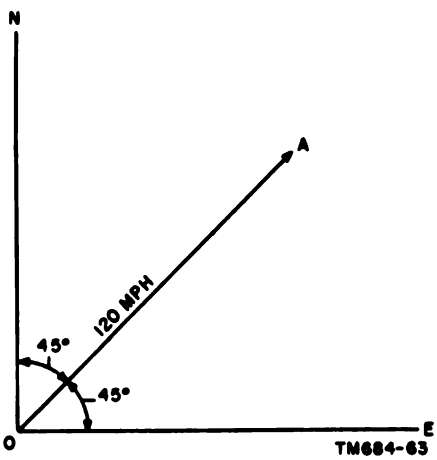


Figure 55. The velocity of an airplane described by a vector.

179. Addition of Vectors, Parallelogram Method

The addition of vectors by the parallelogram method is shown in figure 56. To add vector *OA* to *OC*, draw a vector *OC* with its initial point located at the initial point of vector *OA*, and complete the parallelogram with these vectors forming two sides. The diagonal vector *OB*, with its initial point at the same initial point of *OA* and *OC* and its terminal point at the opposite vertex of the parallelogram, is the sum of *OA* and *OC*. Thus, two vectors (*OA* and *OC*) acting simultaneously on a point or object may be replaced by a single vector called the *resultant* (*OB*). The resultant vector will pro-

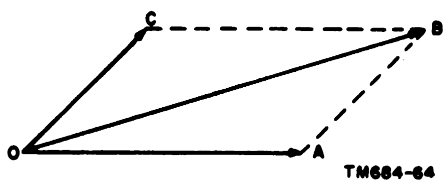


Figure 56. Adding vectors, parallelogram method.

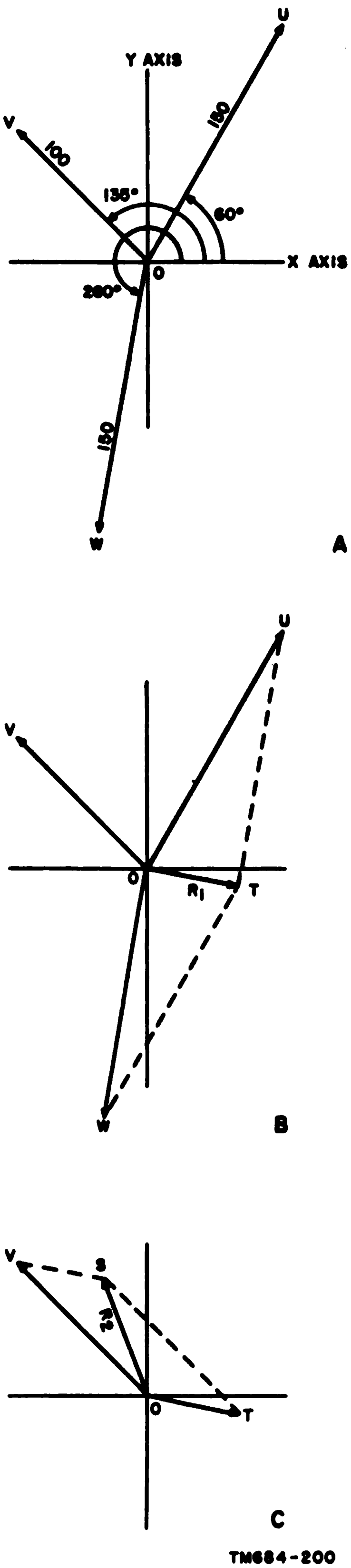


Figure 57. Resolution of three vectors.

duce the same effect on the object as the joint action of the two vectors.

180. Addition of More Than Two Vectors

a. In determining the resultant (par. 179) of vectors when more than two quantities are represented, proceed as follows:

(1) Find the resultant of two of the vector quantities,

(2) Determine the final resultant between the third quantity and the resultant obtained from (1), above.

b. Assume three forces U, V, and W are acting on point O as shown in A, figure 57. Force U exerts 150 pounds at an angle of 60°, V exerts 100 pounds at an angle of 135°, and W exerts 150 pounds at an angle of 260°. Find the resultant of forces on point O.

(1) The resultant of any two vectors, such as U and W, are determined graphically by the line R_1 (B, fig. 57). To solve this problem first draw the vectors to scale at the designated angles; then construct the parallelogram OUTW with adjacent sides WT and UT. The resultant R_1 of OW and OU will be the diagonal OT.

(2) Combine the resultant R_1 with force V, then construct another parallelogram to scale as in (1), above. The final resultant R_2 is similarly determined by the line SO (C, fig. 57).

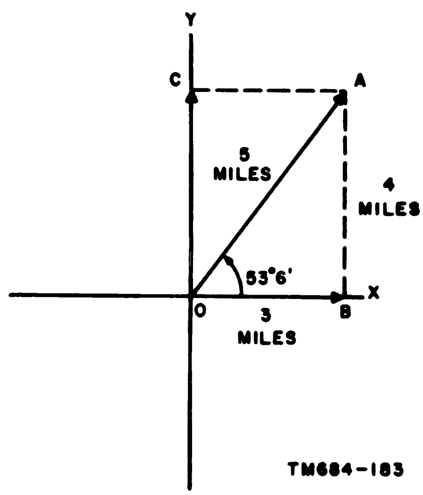


Figure 58. Horizontal and vertical components of vector.

This, then, is the resolution of all three forces U, V, and W acting on point O.

181. Components of a Vector

a. A vector may be resolved into components along any two specified directions. If the directions of the components are chosen so that they are at right angles to each other, the components are called *rectangular components.*

b. By placing the initial point of a vector at the origin of the X and Y axes, the rectangular components are readily obtained either graphically or by computation. In figure 58, a vector with a magnitude of 5 and a direction of 53° 6′ is shown broken down into a horizontal component of 3 and a vertical component of 4. This is done by using the sine and cosine function as follows:

$$\sin 53^\circ\ 6' = \frac{BA}{5}$$

$$.79968 = \frac{BA}{5}$$

$$BA = 5 \times .79968$$

$$= 4 \text{ (approx)}$$

$$\cos 53^\circ\ 6' = \frac{OB}{5}$$

$$.60042 = \frac{OB}{5}$$

$$OB = 5 \times .60042$$

$$= 3 \text{ (approx)}$$

PART II

APPLICATIONS OF MATHEMATICAL PRINCIPLES TO COMMON COMMUNICATIONS-ELECTRONICS PROBLEMS

CHAPTER 13

INTRODUCTION

182. Series Circuits

In a *series circuit*, electrical energy is supplied to a number of devices in series; that is the same current passes through each device in completing its path to the source of supply. Figure 59 shows a resistance, an inductor, and a capacitor connected in series with a voltage source.

a. The current is the same in all parts of a series circuit.

b. The total voltage drop (E_t) in a series circuit is equal to the sum of the voltage drops across individual loads:

$$E_t = E_1 + E_2 + E_3 + \dots$$

c. The total resistance (R_t) of a series circuit is equal to the sum of all individual resistance:

$$R_t = R_1 + R_2 + R_3 + \dots$$

d. The total inductance L_t of a series circuit is equal to the sum of the individual inductances:

$$L_t = L_1 + L_2 + L_3 + \dots$$

e. The reciprocal of the total capacitance (C_t) is equal to the sum of the reciprocals of the separate capacitances. The total capacitance is also less than the capacitance of any one of the capacitors, and is expressed as follows:

$$\frac{1}{C_t} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

If only two capacitances are in series, a simplified formula can be derived by combining fractions over an LCD, and taking the reciprocal:

$$\frac{1}{C_t} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C_t} = \frac{C_2}{C_1C_2} + \frac{C_1}{C_1C_2}$$

$$\frac{1}{C_t} = \frac{C_1 + C_2}{C_1C_2}$$

$$C_t = \frac{C_1C_2}{C_1 + C_2}$$

If two or more capacitors of equal value are placed in series, the total capacitance is equal to the value of one capacitor (c) divided by the number of capacitors used (n):

$$C_t = \frac{C}{n}$$

This equation can be derived as follows (assuming 3 equal-value capacitors):

$$\frac{1}{C_t} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$\frac{1}{C_t} = \frac{1}{C} + \frac{1}{C} + \frac{1}{C}$$

$$\frac{1}{C_t} = \frac{3}{C}$$

$$C_t = \frac{C}{3}$$

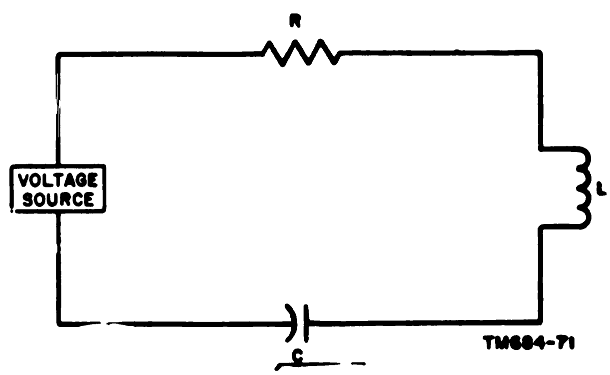


Figure 59. Example of a series circuit.

183. Parallel Circuits

Figure 60 is an example of a simple parallel circuit, with two resistors connected in parallel across a generator. As indicated by the arrows, the current from the generator separates into two parts, each resistor receiving a part of the total current. The larger fraction of current flows through the branch of less resistance, and the smaller fraction of current flows through the branch of greater resistance. The two parts of the current join again upon leaving the resistors.

a. The total current (I_t) in a parallel circuit is the sum of the currents in the separate branches:

$$I_t = I_1 + I_2 + I_3 + \ldots\ldots\ldots\ldots$$

b. The voltage (E) across each branch of a parallel circuit is the same:

$$E_1 = E_2 = E_3 \ldots\ldots\ldots$$

c. The reciprocal of the total resistance (R_t) of all resistors in a parallel circuit is equal to the sum of the reciprocals of the separate resistance. The total resistance is also less than the resistance of any one of the resistors, and is expressed as follows:

$$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots\ldots$$

If only two resistors are in parallel, a simplified formula can be derived for the total resistance as for total capacitance in a series circuit (par. 182*e*):

$$R_t = \frac{R_1 R_2}{R_1 + R_2}$$

If two or more resistors of the same value are placed in parallel, the total resistance is equal to the value of one resistor (R) divided by the number of resistors used (n), as for capacitances, in series (par. 182*e*):

$$R_t = \frac{R}{n}$$

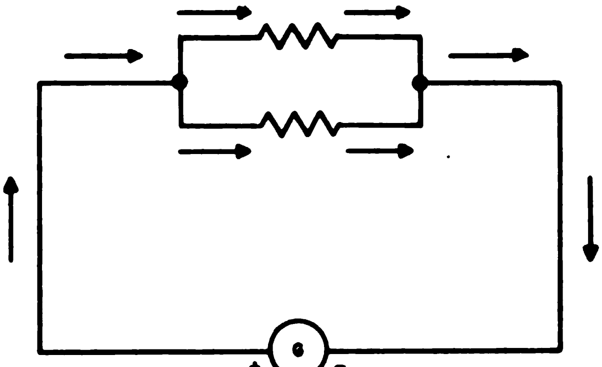


Figure 60. Example of a parallel circuit.

d. The reciprocal of the total inductance (L_t) in a parallel circuit is equal to the sum of the reciprocals of the separate inductances, as with resistances (*c* above):

$$\frac{1}{L_t} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots.$$

The rules covering the calculation of resistances in parallel (*c* above) also apply to inductances in parallel.

e. The total capacitance in a parallel circuit is equal to the sum of the individual capacitances, as for resistances and inductances in series (par. 182*c* and *d*):

$$C_t = C_1 + C_2 + C_3 + \ldots.$$

The rules covering the calculation of resistances and inductances in series also apply to capacitances in parallel.

184. Series-Parallel Circuit

A series-parallel circuit is simply a combination of a series circuit and a parallel circuit. The rules covering series circuits (par. 182) apply to the series portion of the circuit, and the rules covering parallel circuits (par. 183) apply to the parallel portion of the circuit. The examples given in chapters 14 through 18 more clearly illustrate the various types of circuits.

CHAPTER 14

PROBLEMS IN DC ELECTRICITY

185. General

In circuits using constant-value dc electricity, only the effects of the resistance in the circuit are significant, because inductance and capacitance depend on varying current or voltage. Consequently, the examples given in this chapter involve only resistances.

186. Ohm's Law

a. An important relationship between current (I), voltage (E), and resistance (R) in a circuit is given by Ohm's law which states that the current in an electrical circuit varies directly as the voltage and inversely as the resistance. Expressed in a formula, the relationship is:

$$I = \frac{E}{R}$$

The formula may also appear in the following forms:

$$E = IR$$

$$R = \frac{E}{I}$$

b. The following example illustrates Ohm's law:

Example: Solve the following problem: A voltmeter (voltage measuring device) connected directly across a resistance reads 65 volts (fig. 61). An ammeter (current measuring device) connected in series reads 5.3 amperes. What is the value of the resistance in ohms?

$$E = 65, I = 5.3, R = ?$$

$$R = \frac{E}{I}$$

$$R = \frac{65}{5.3}$$

$$= 12.26 \text{ or } 12.3 \text{ ohms.}$$

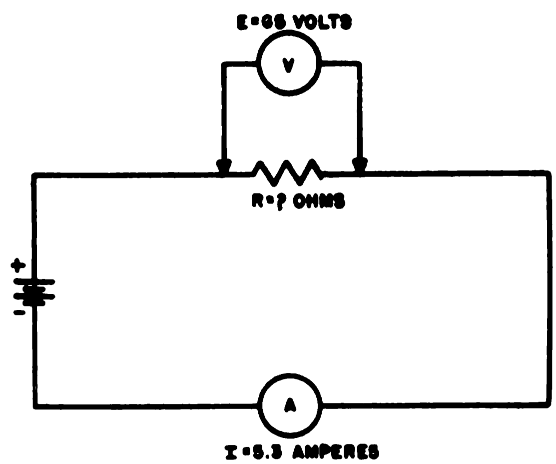


Figure 61. Simple circuit with unknown resistance.

187. Solving Series Circuits

The following example illustrates the method of using Ohm's law and the principles of series circuits (par. 182) to solve series dc circuits.

Example: Solve the following problem: Resistors R_1, R_2, and R_3 are connected in series across a 110-volt generator (fig. 62). If resistor $R_1 = 6.5$ ohms, resistor $R_2 = 10.3$ ohms, and resistor $R_3 = 7.6$ ohms, what is the total current in the circuit? What is the voltage drop across each resistance?

Step 1. Find the total resistance in the circuit.

$$R_t = R_1 + R_2 + R_3$$
$$= 6.5 + 10.3 + 7.6$$
$$= 24.4 \text{ ohms total resistance}$$

Step 2. Find the total current in the circuit.

$$E = IR$$
$$110 = I(24.4)$$
$$24.4I = 110$$
$$I = 4.508 \text{ amperes total current}$$

Step 3. Find the voltage drop across R_1.

$$E = IR$$
$$= 4.508(6.5)$$
$$= 29.302 \text{ volts across } R_1$$

Step 4. Find the voltage drop across R_2.

$$E = IR$$
$$= 4.508(10.3)$$
$$= 46.432 \text{ volts across } R_2$$

Step 5. Find the voltage drop across R_3.

$$E = IR$$
$$= 4.508(7.6)$$
$$= 34.261 \text{ volts across } R_3$$

Check: $34.261 + 46.432 + 29.303 = 109.996$ or 110 volts.

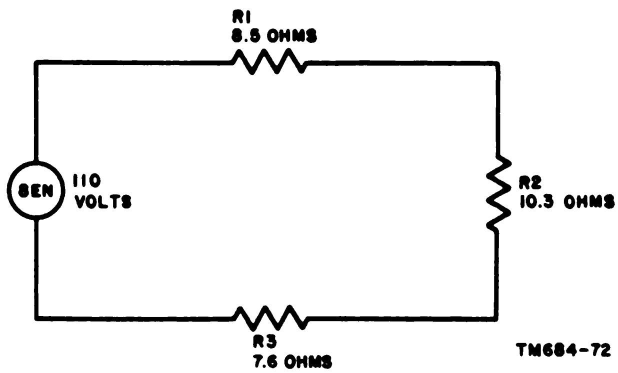


Figure 62. Series circuit with unknown current.

188. Solving Parallel Circuits

The following example illustrates the method of using Ohms' law and the principles of parallel circuits (par. 183) to solve parallel dc circuits.

Example: Solve the following problem:

In figure 63, a resistor of 200 ohms (R_1), a resistor of 600 ohms (R_2), and an unknown resistor (R_3) are connected in parallel across a source of emf. The voltage across R_1 is 40 volts. The current through the resistor of unknown value (R_3) is 0.40 ampere. Find (a) the value of R_3, (b) the total resistance of the circuit, and (c) the total current, in the circuit.

Step 1. Find the voltage across R_3.

$$E_1 = E_2 = E_3$$

Since the voltage across R_1 is 40 volts, the voltage across R_3 is also 40 volts.

Step 2. Find the resistance of R_3.

$$R_3 = \frac{E_3}{I_3}$$
$$= \frac{40}{.4}$$
$$= 100 \text{ ohms}$$

Step 3. Find the total resistance of the three resistors.

$$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$
$$\frac{1}{R_t} = \frac{1}{200} + \frac{1}{600} + \frac{1}{100}$$
$$\frac{1}{R_t} = \frac{3}{600} + \frac{1}{600} + \frac{6}{600}$$
$$\frac{1}{R_t} = \frac{10}{600}$$
$$10R_t = 600$$
$$R_t = \frac{600}{10}$$
$$= 60 \text{ ohms}$$

Step 4. Find the line current in the circuit.

$$I_t = \frac{E_t}{R_t}$$
$$= \frac{40}{60}$$
$$= 0.667 \text{ ampere}$$

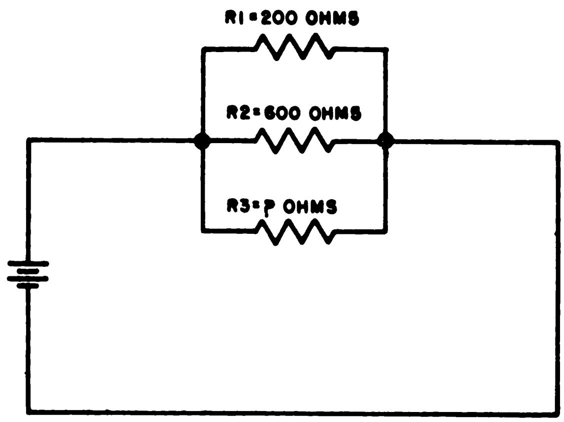


Figure 63. Parallel circuit with three resistances, one unknown.

189. Solving Series-Parallel Circuits

A simple series-parallel circuit, with series-connected resistors R_2 and R_3 connected in parallel with resistor R_4 and the combination connected in series with resistors R_1 and R_5, is shown in A, figure 64. The following example uses B through D, figure 64, to illustrate the method of solving series-parallel dc circuits.

Example: Find the current through each resistance and the voltage drop across each resistance in A, figure 64.

Step 1. Since R_2 and R_3 are in series, their total resistance is the sum (B, fig. 64) of the two resistances.

$$R_{2,3} = R_2 + R_3$$
$$= 5 + 15$$
$$= 20 \text{ ohms}$$

Step 2. $R_{2.3}$ is in parallel with R_4. Find the total resistance of the combination (C, fig. 64).

$$\frac{1}{R_{2.3.4}} = \frac{1}{R_{2.3}} + \frac{1}{R_4}$$
$$\frac{1}{R_{2.3.4}} = \frac{1}{20} + \frac{1}{30}$$
$$\frac{1}{R_{2.3.4}} = \frac{3}{60} + \frac{2}{60}$$
$$\frac{1}{R_{2.3.4}} = \frac{5}{60}$$
$$5R_{2.3.4} = 60$$
$$R_{2.3.4} = \frac{60}{5}$$
$$R_{2.3.4} = 12 \text{ ohms}$$

Step 3. $R_1, R_{2.3.4}$ and R_5 are in series. Their total resistance is the sum (D, fig. 64) of the resistances.

$$R_{1.2.3.4.5} = R_1 + R_{2.3.4} + R_5$$
$$= 3 + 12 + 10$$
$$= 25 \text{ ohms}$$

Step 4. Find the total current sent through these resistances by a voltage of 100 volts.

$$I_t = \frac{E_t}{R_t}$$
$$= \frac{100}{25}$$
$$= 4 \text{ amperes}$$

Step 5. Find the voltage drop across $R_{2.3.4}$.

$$E_{2.3.4} = IR_{2.3.4}$$
$$= 4 \times 12$$
$$= 48 \text{ volts}$$

Step 6. Analyze the parallel circuit.

The voltage across R_4 is 48 volts. Find the current.

$$I_4 = \frac{E_4}{R_4}$$
$$= \frac{48}{30}$$
$$= 1.6 \text{ amperes}$$

The voltage across R_2 and R_3 also is 48 volts, and the resistance $R_{2.3}$ is 20 ohms. Find the current.

$$I_{2.3} = \frac{E_{2.3}}{R_{2.3}}$$
$$= \frac{48}{20}$$
$$= 2.4 \text{ amperes } (I_2 = I_3)$$

Step 7. Find all voltage drops.

$$
\begin{aligned}
E_1 &= I_1R_1\\
&= 4 \times 3\\
&= 12 \text{ volts}\\
E_2 &= I_2R_2\\
&= 2.4 \times 5\\
&= 12 \text{ volts}\\
E_3 &= I_3R_3\\
&= 2.4 \times 15\\
&= 36 \text{ volts}\\
E_4 &= I_4R_4\\
&= 1.6 \times 30\\
&= 48 \text{ volts}\\
E_5 &= I_5R_5\\
&= 4 \times 10\\
&= 40 \text{ volts}
\end{aligned}
$$

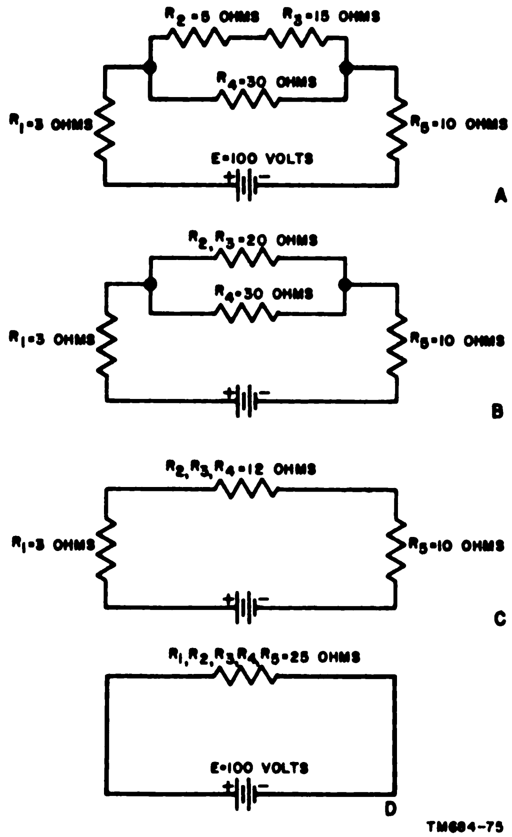


Figure 64. Solving a series-parallel circuit.

190. Solving More Complex Electrical Problems by Using Kirchhoff's Laws

a. General. The more complex series-parallel problems are often more readily solved by using Kirchhoff's laws. A full treatment of the electrical phenomena embodied in Kirchhoff's laws is not within the scope of this manual. For a complete treatment of electrical theory on this subject, see TM 11–661. The basic principles of Kirchhoff's laws are as follows:

(1) The algebraic sum of the currents at any junction of conductors is zero.

(2) The algebraic sum of the electromotive forces and voltage drops around a closed circuit is zero.

b. Understanding Kirchhoff's Laws. The first of Kirchhoff's laws simply means that there is just as much current flowing away from a point as there is flowing to it. The second law simply means that the voltage source is equal to the sum of the voltage drops around any closed circuit. For example, starting at point X (fig. 65) and going around the circuit clockwise, the following equation is obtained:

$$E - IR_1 - IR_2 - IR_3 = 0$$

Substituting the values of resistance as indicated in the figure, the equation becomes:

$$37 - 13I - 9I - 11I = 0$$

Collecting like terms and solving for I gives:

$$
\begin{aligned}
37 - 33I &= 0\\
33I &= 37\\
I &= 1.121 \text{ amperes}
\end{aligned}
$$

To prove that this is correct, use Ohm's law as follows:

$$
\begin{aligned}
E_1 &= IR_1 = 1.121 \times 13 = 14.58 \text{ volts}\\
E_2 &= IR_2 = 1.121 \times 9 = 10.09 \text{ volts}\\
\underline{E_3} &\underline{= IR_3 = 1.121 \times 11 = 12.33 \text{ volts}}\\
E_t &= IR_t = 1.121 \times 33 = 37.00 \text{ volts}
\end{aligned}
$$

Thus, the sum of the voltage drops equals the applied voltage and the second law is verified.

c. Solving Series-Parallel Circuits Using Kirchhoff's Laws. Problems involving series-parallel circuits are readily solved by using Kirchhoff's laws and simultaneous equations (par. 84). The example below illustrates such a problem.

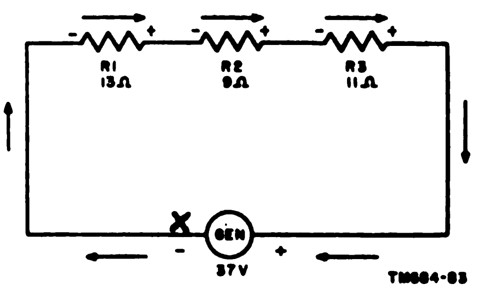


Figure 65. Example of Kirchhoff's second law.

Example: Solve for the current in each branch of the circuit shown in figure 66.

Step 1. Assume a direction for the current flow in each branch, as shown in the figure. (It will be shown that the direction assumed does not affect the accuracy of the result.) According to Kirchhoff's first law, the current I_1 flowing through the 6-ohm resistor plus the current I_2 flowing through the 7- and 8-ohm resistors equals the current $I_1 + I_2$ flowing through the remainder of the circuit, which includes the 5-ohm resistor.

Step 2. Considering the first part of the circuit, from point B through the generator and around the circuit back to point B through the 6-ohm resistor, the application of Kirchhoff's second law yields the following equation:

$$10 - 5(I_1 + I_2) - 6I_1 = 0$$
$$10 - 5I_1 - 5I_2 - 6I_1 = 0$$
$$10 - 11I_1 - 5I_2 = 0 \text{ (equation 1).}$$

Step 3. Considering the path from point B through the generator and through points A, X, and Y back to B, the application of Kirchhoff's second law yields the following equation:

$$10 - 5(I_1 + I_2) - 7I_2 - 8I_2 = 0$$
$$10 - 5I_1 - 5I_2 - 7I_2 - 8I_2 = 0$$
$$10 - 5I_1 - 20I_2 = 0 \quad \text{(equation 2).}$$

Step 4. Using the methods of solving simultaneous equations described in paragraph 116, solve for I_1 by multiplying equation 1 by 4 and subtracting equation 2 from the new equation:

$$40 - 44I_1 - 20I_2 = 0$$
$$\underline{10 - 5I_1 - 20I_2 = 0}$$
$$30 - 39I_1 = 0$$
$$-39I_1 = -30$$
$$I_1 = 0.769 \text{ ampere}$$

Step 5. Solve for I_2 by substituting the value of I_1 in either equation 1 or equation 2, or by eliminating I_1 in solving the simultaneous equations. Substituting I_1 in equation 2 yields the following:

$$10 - 5(0.769) - 20I_2 = 0$$
$$10 - 3.845 - 20I_2 = 0$$
$$-20I_2 = -6.155$$
$$I_2 = 0.308 \text{ ampere}$$

Step 6. The current in the left-hand side of the circuit is $I_1 + I_2$ or 1.077 amperes.

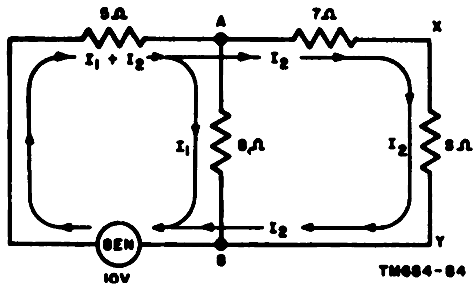


Figure 66. Solving series-parallel circuits, using Kirchhoff's laws.

d. Direction of Current Flow. If the direction of current flow is assumed incorrectly, the computed value for the current will have a negative sign; however, the magnitude of the current will be the same. Therefore, to correct the error, simply reverse the assumed direction of current flow on the diagram.

e. Facts to Remember When Working Problems. The solution of problems involving series-parallel circuits by the above method normally is relatively simple. The important facts to remember when working such problems are:

(1) Assume any direction of current flow in the beginning.

(2) Take any path around any portion of the circuit, as long as the path is a complete circuit.

(3) Observe the polarities of the circuit, both voltage sources and voltage drops.

(4) Be sure to have as many equations as there are unknowns.

191. Dc Power

In dc circuits, the amount of power absorbed by a resistor or the resistance of a circuit is easily determined by Joule's law:

$P = I^2R$, where:

P = power absorbed in watts

I = total current in amperes

R = total resistance of the circuit in ohms

Since the voltage drop (E) across a resistor (R) is equal to IR, the formula above may also be written: $P = IR \times I = EI$.

Example 1: Find the power consumed in a 50-ohm resistor when a current of 5 amperes flows through it.

$$P = I^2R = 5^2 \times 50 = 1{,}250 \text{ watts}$$

Example 2: Find the power delivered by a 12-volt battery when the current drain is 6 amperes.

$$P = EI = 12 \times 6 = 72 \text{ watts}$$

192. Review Problems—Dc Electricity

a. (1) The resistance of a tungsten lamp is 20 ohms when the lamp is cold. What current will the lamp draw the instant it is placed across a 110-volt line? (2) When the lamp is glowing at full brilliancy, its resistance rises to 84 ohms. What is the final steady current of the lamp?

b. An adjustable resistor has a minimum setting of 14 ohms and a maximum setting of 50 ohms. (1) What ranges of resistance can be covered with two of these resistors connected in series? (2) What ranges of resistance can be covered with two of these resistors connected in parallel?

c. When a 6,500-ohm resistance is connected into the plate circuit of a radio tube, the plate current is 34 milliamperes. (1) What is the voltage drop across the 6,500-ohm resistance? (2) How much power is consumed by the resistor?

d. Three resistors of 20 ohms, 30 ohms, and 50 ohms, respectively, are connected in series. The current through R_1 (20 ohms) is 0.8 ampere. (1) What is the current through R_3 (50 ohms)? (2) What is the voltage across R_2 (30 ohms)? (3) What is the total voltage drop across the three resistors?

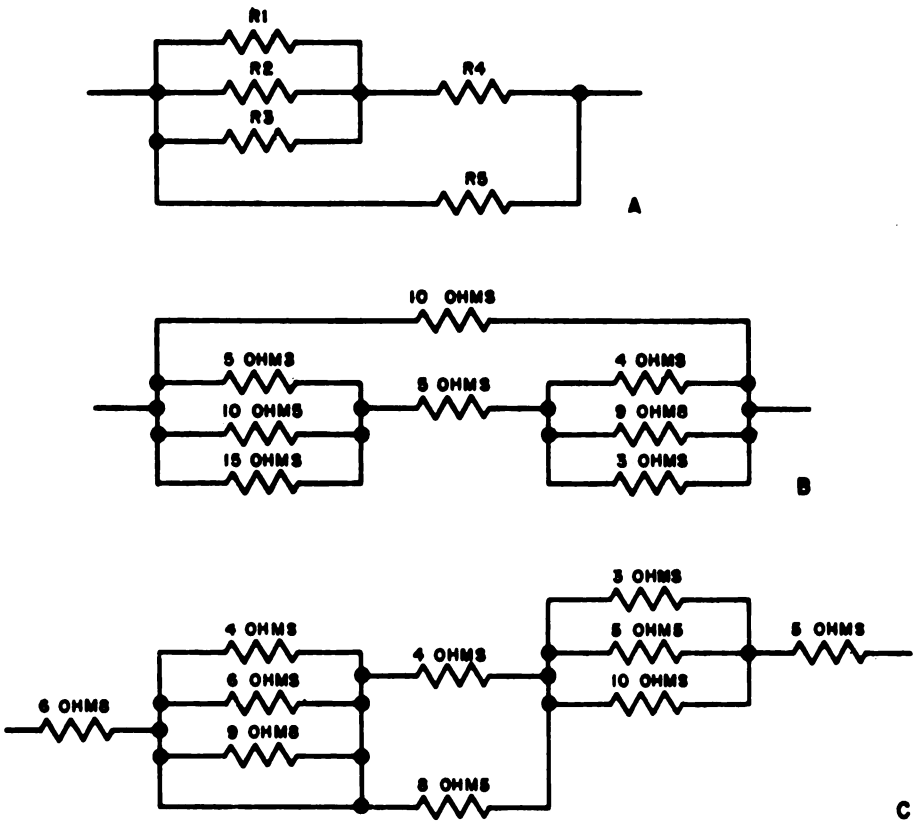
e. A divided circuit has three branches of 5, 10, and 20 ohms resistance, respectively. (1) What is the joint conductance of the three branches? (Conductance is the reciprocal of resistance.) (2) What is the joint resistance? (3) A current of 20 amperes flows in the 5-

ohm branch; find the current in each of the other branches. (4) Find the combined current.

f. A parallel circuit has branches with resistances of 1, 3, 10, 20, and 50 ohms, respectively. (1) What is the conductance of each branch? (2) What is the conductance of the combination? (3) What is the resistance of the combination?

g. Three resistors R_1 (36 ohms), R_2 (42 ohms) and R_3 are connected in series with a generator. An ammeter inserted in the circuit between R_1 and R_2 reads 2.4 amperes, and a voltmeter across R_3 reads 41 volts. (1) What is the resistance of R_3? (2) What is the voltage across R_1? (3) What is the voltage across R_2? (4) What is the voltage across the generator?

h. Find the total resistance of: (1) circuit *A* when $R_1 = 6$ ohms, $R_2 = 9$ ohms, $R_3 = 17$ ohms, $R_4 = 5$ ohms, $R_5 = 11$ ohms; (2) circuit a when $R_1 = 12$ ohms, $R_2 = 25$ ohms, $R_3 = 19$ ohms, $R_4 = 8$ ohms, $R_5 = 12$ ohms. (3) circuit *B;* (4) circuit *C*.



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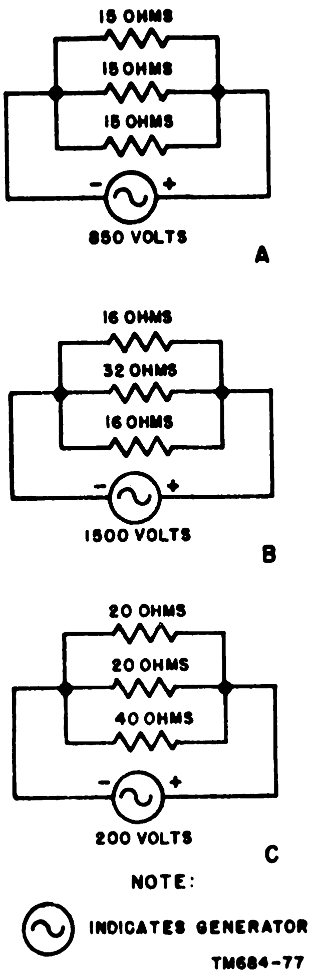
i. A 10-ohm resistor is connected in series with a 15-ohm resistor. (1) What voltage must be placed across the two resistors to send a current of 5 amperes through it? (2) What would the voltage be across each resistor?

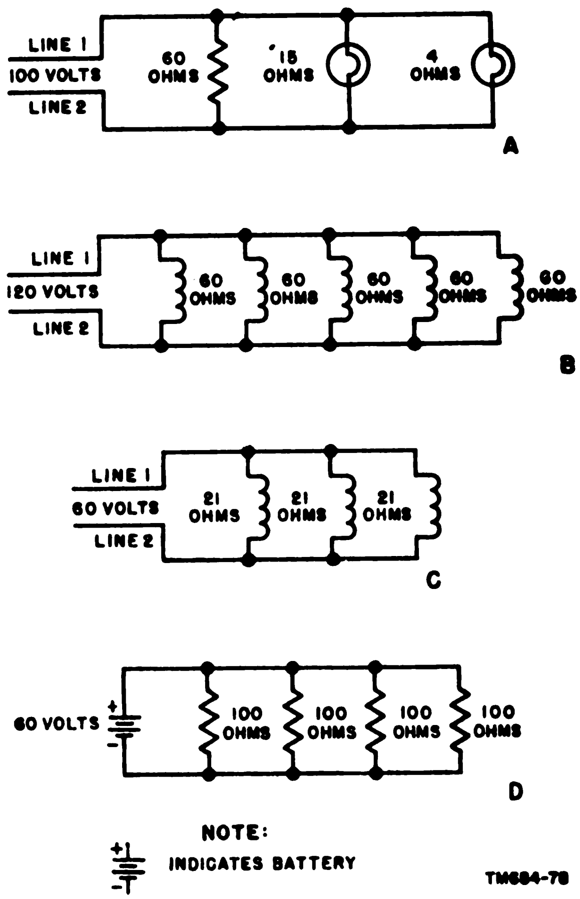
j. (1) What voltage is required to force a current of 10 amperes through a parallel combination of three branches having resistances of 15.3 ohms, 1.3 ohms, and 10.5 ohms, respectively? (2) What will the current be in each branch? (3) What is the voltage drop across each branch?

k. A generator has an output voltage of 110 volts. (1) What current is flowing in a wire of 0.02 ohm connected across the terminals? (2) What current will flow if an incandescent lamp of 484 ohms is also connected across the generator?

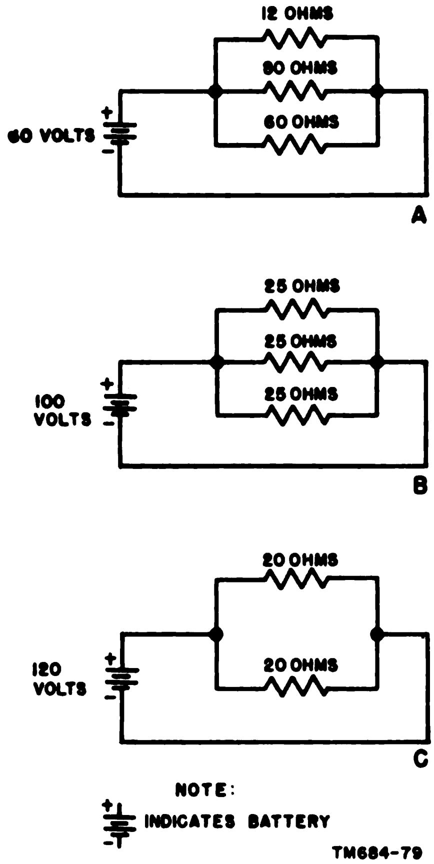
l. Find the total resistance of each of the parallel circuits A, B, and C.



m. Find the total resistance of each of the parallel circuits A, B, C, and D.

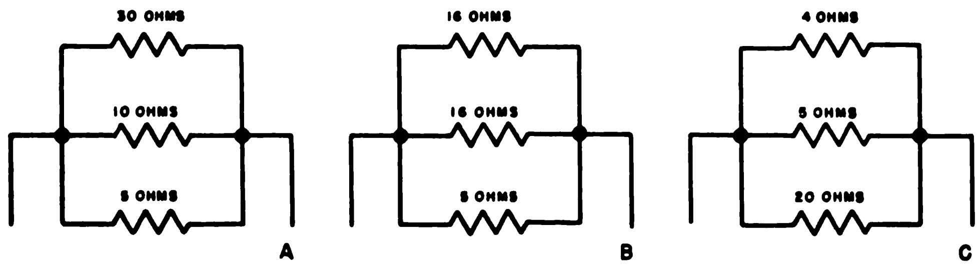


n. Find the total resistance of each of the parallel circuits A, B, and C.



TM684-79

o. Find the total resistance of each of the parallel circuits A, B, and C.



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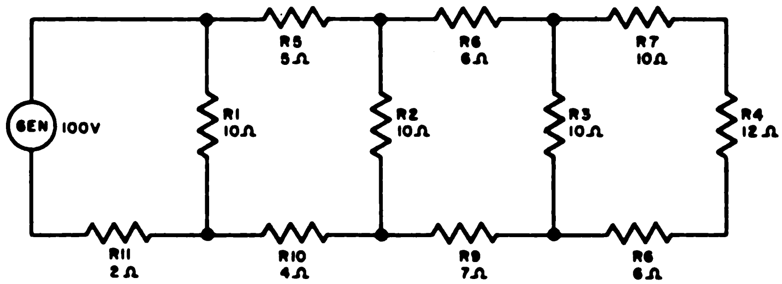
p. Find the current through each resistor in the circuit below.



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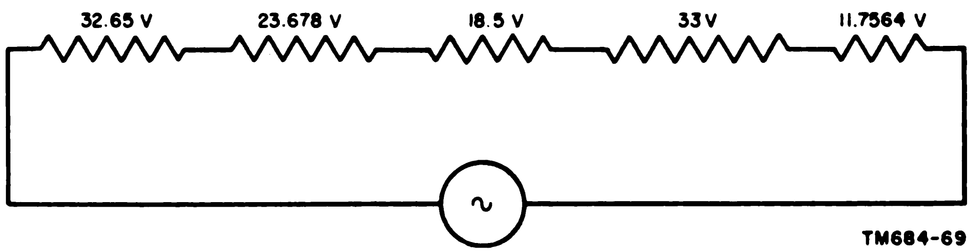
q. Find the total resistance in the circuit below when a current of .5 amperes flows through it.



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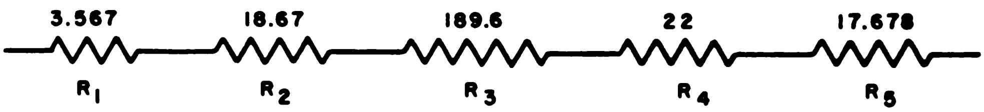
r. Find the current through the resistors in the circuit below when 115 volts is applied across the circuit.



TM684-68

CHAPTER 15

PROBLEMS IN AC ELECTRICITY

193. General

In circuits using ac electricity[1], the current is affected by inductance and capacitance as well as resistance. In addition, certain combinations of these loads will produce unusual effects, such as resonance (par. 202), not experienced in dc circuits. These phenomena are used extensively in electrical and electronic circuits. Consequently, problems in ac electricity are more complex than corresponding problems in dc electricity.

194. Application of Vectors and Trigonometry in Solving Ac Circuit Problems

a. As discussed in chapter 12, a vector is a line whose length and direction represent accurately a given quantity; the quantity thus represented is a *vector quantity*. Because the magnitude of ac currents and voltages varies from instant to instant, the magnitude is a function of time, and the current and voltage can be expressed as vectors: The length of the vector represents the magnitude of the current or voltage, and the direction represents its relationship in time to another vector (*b* below).

b. When a circuit contains inductance or capacitance, the current in the circuit is not in phase with the voltage that produces it. In other words, the instant the voltage is zero, the current that it produces has a value other than zero, or when the voltage is at its maximum, the current has a value different from its maximum value. The current is said to *lead* the voltage if the current reaches its maximum *before* the voltage maximum occurs; the current is said to *lag* the voltage if the current reaches its maximum *after* the voltage maximum occurs. The relationship between current and voltage can be represented by vectors, with one vector representing current, another voltage, and with the angle between them indicating the amount of lag or lead. Figure 67 shows a vector representation of leading and lagging current. The angle is called the phase angle.

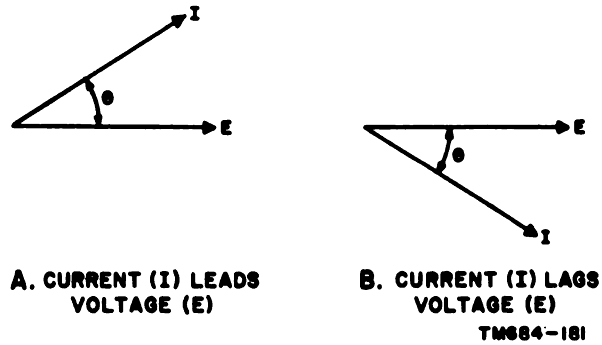


Figure 67. Vector representation of leading and lagging current.

c. The voltage drop across a resistor also may be represented by a vector having the same direction as the vector representing the current flowing through the resistor. In other words, the voltage across the resistor and the current flowing through it are in phase.

d. The voltage drop across a capacitor may be represented by a vector making an angle of 90° with the vector representing the current flowing through the capacitor. In a purely capacitive circuit, the current will lead the applied voltage by an angle of 90°.

e. The voltage drop across an inductor may be represented by a vector making an angle of 90° with the vector representing the current flowing through the inductance. In a purely inductive circuit, the current will lag the applied voltage by an angle of 90°.

[1] This chapter is limited to the application of mathematics to single-phase, sinusoidal ac. The electrical phenomena of this type of ac are treated briefly. See TM 11-681 for a complete treatment of single-phase, sinusoidal ac.

f. In a circuit that contains inductance, capacitance, and resistance, the current will lead or lag the applied voltage by a phase angle of less than 90°.

g. The example below illustrates the use of vectors in the solution of a typical ac circuit problem. Paragraphs 199 through 201 give a more detailed coverage of problems of this type.

Example: In a series circuit (fig. 68), the voltage drop across the capacitor (E_C) is 10 volts, the voltage drop across the inductance (E_L) is 50 volts, and the voltage drop across the resistance (E_R) is 30 volts. Determine the magnitude of the applied voltage. By what phase angle (A) does the current lead or lag the applied voltage in the circuit?

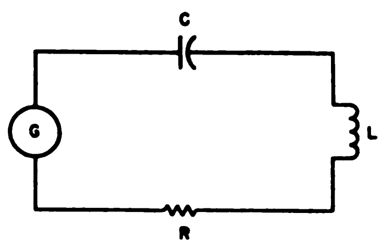


Figure 68. An ac series circuit containing inductance, capacitance, and resistance.

Step 1. The vector diagram for this circuit is shown in figure 69. In a series circuit, the same current flows through each element. Draw the vector representing the current (I) in a horizontal position. The angles of all vectors representing voltage drops are given with respect to the current.

Step 2. Draw the vector E_L, representing the voltage drop across the inductance, at an angle of 90° with the vector I.

Step 3. Draw the vector E_C, representing the voltage drop across the capacitor, at a angle of —90° with the vector I.

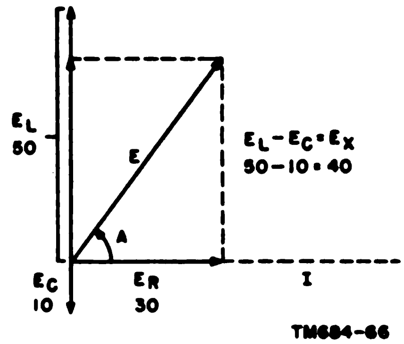


Figure 69. A vector diagram of an ac series circuit containing inductance, capacitance, and resistance.

Step 4. The vector E_R, representing the voltage drop across the resistor, has the same direction as the vector I.

Step 5. The vector sum of these voltage drops is equal to the applied voltage.

Along the horizontal:

$E_L = 0,\ E_C = 0,\ E_R = 30$

Along the vertical:

$E_L = 50,\ E_C = -10,\ E_R = 0$

Step 6. Adding the horizontal and vertical voltage drops, respectively:

$E_L + E_C + E_R = 0 + 0 + 30 = 30$

$E_L + E_C + E_R = 50 + (-10) + 0 = 40$

Step 7. Because the vectors form a right triangle, with the applied voltage E as the hypotenuse and E_R and E_X as the sides (fig. 69), the law of right triangles (par. 133) can be used to solve for one of the quantities when the other two are known. From this law, the relationship between E, E_R, and E_X is expressed by the formula

$E = \sqrt{E_R^2 + E_X^2}$.

$$\begin{aligned} E &= \sqrt{E_R^2 + E_X^2} \\ &= \sqrt{(30)^2 + (40)^2} \\ &= \sqrt{900 + 1600} \\ &= \sqrt{2500} \\ &= 50 \text{ volts} \end{aligned}$$

Step 8. The formula for determining angle A which the vector representing the applied voltage makes with the vector I (fig. 69) is $\tan A = \frac{E_X}{E_R}$.

$$\tan A = \frac{E_X}{E_R}$$
$$= \frac{40}{30} \text{ or } \frac{4}{3}$$
$$= 1.33333$$
$$A = 53° \, 7' \, 48''$$

Step 9. The circuit is predominately inductive; therefore, the current lags the applied voltage by a phase angle of 53° 7′ 48″.

195. Ohm's Law Applied to Ac Circuits

Because of the effects of inductance and capacitance in ac circuits, Ohm's law (par. 186) must be modified to take these added effects into consideration.

a. If the circuit contains a combination of resistance and inductive reactance (par. 196) or capacitive reactance (par. 197), or both, the overall effect is called impedance (par. 198), and Ohm's law is modified to read:

$$I = \frac{E}{Z}$$

where I is the current in amperes, E the ac voltage in volts, and Z the impedance in ohms. This formula may also be written:

$$E = IZ$$
$$Z = \frac{E}{I}$$

b. If the circuit contains reactances only, the formulas become:

$$I = \frac{E}{X}$$
$$E = IX$$
$$X = \frac{E}{I}$$

where X is the total reactance (par. 198*a*) of the circuit in ohms.

c. If the circuit contains resistance only, the formula is the same as in a dc circuit (par. 186).

d. The application of these formulas in solving ac circuit problems is covered in paragraphs 196 through 203.

196. Inductive Reactance

Inductance enables an electric circuit to build up a voltage by electromagnetic induction whenever the current strength changes. The induced voltage always opposes the applied voltage and thus retards the change in the current. *Inductive reactance* is the effect of inductance expressed in *ohms*. The formula for finding inductive reactance is:

$$X_L = 2\pi fL$$

where X_L is the inductive reactance in ohms, L is the inductance in henrys, and f is the frequency in cps.

Example 1: Determine the inductive reactance of a coil if the ac in the circuit has a frequency of 100 cps, and the inductance of the coil is 0.036 henry.

$$X_L = 2\pi fL$$
$$= 2 \times 3.14 \times 100 \times .036$$
$$= 628 \times .036$$
$$= 22.608 \text{ ohms}$$

Example 2: If a coil with an inductance of 0.2 henry and negligible resistance is connected across the terminals of a 220-volt, 60-cycle ac generator, how much current will flow through the coil?

Step 1. Find the inductive reactance of the coil.

$$X_L = 2\pi fL$$
$$= 2 \times 3.14 \times 60 \times .2$$
$$= 376.8 \times .2$$
$$= 75.36 \text{ ohms}$$

Step 2. Find the amount of current that will flow through the coil.

$$I = \frac{E}{X_L}$$
$$= \frac{220}{75.36}$$
$$= 2.92 \text{ amperes}$$

197. Capacitive Reactance

Capacitance enables a capacitor to retain an electric charge which opposes any changes in the voltage of the circuit in which the capacitor is connected. *Capacitive reactance* is the effect of the capacitance expressed in *ohms*.

The formula for finding capacitive reactance is:

$$X_C = \frac{1}{2\pi fC}$$

where X_C is the capacitive reactance, C is the capacitance expressed in farads, and f is the frequency in cycles per second.

Example: A 110-volt, 60-cycle ac generator is connected in series with a 1-microfarad (10^{-6} farad) capacitance. What is the capacitive reactance of the circuit?

$$\begin{aligned} X_C &= \frac{1}{2\pi fC} \\ &= \frac{1}{2 \times 3.14 \times 60 \times 10^{-6}} \\ &= \frac{10^6}{6.28 \times 60} \\ &= \frac{1{,}000{,}000}{376.8} \\ &= 2{,}653 \text{ ohms} \end{aligned}$$

198. Impedance

a. The *impedance* of a circuit is the circuit's total opposition to the flow of current. In a dc circuit, the opposition consists of resistance alone. In an ac circuit, the opposition consists of resistance and reactance (X). Inductive and capacitive reactances can be combined, but because their effects in the circuit are exactly opposite—inductive reactance causes the current to lag the voltage by 90° and capacitive reactance causes the current to lead the voltage by 90°—they are combined by subtraction:

$X = X_L - X_C$ or

$X = X_C - X_L$ (subtracting the smaller from the larger)

b. Resistance and reactance cannot be added directly, but they can be considered as two vectors acting at right angles to each other. Thus, the relation between resistance, reactance, and impedance may be illustrated by a right triangle (fig. 70). Since these quantities may be related to the sides of a right triangle, the formula for finding the impedance of a circuit is:

$$Z^2 = R^2 + X^2 \text{ or } Z = \sqrt{R^2 + X^2}$$

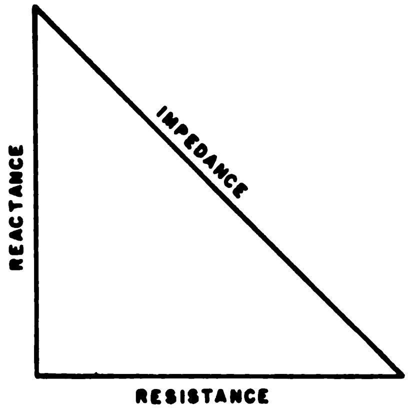


Figure 70. The resistance-reactance-impedance triangle.

where Z is the impedance in ohms, R is the resistance in ohms, and X is the reactance in ohms.

Example 1: A 110-volt, 60-cycle ac generator is connected in series with a 1-microfarad capacitance and a 1,000-ohm resistance. The capacitive reactance of the circuit is 2,650 ohms. What is the impedance of the circuit?

$$\begin{aligned} Z &= \sqrt{R^2 + XC^2} \\ &= \sqrt{(1000)^2 + (2650)^2} \\ &= \sqrt{(10^3)^2 + (2.65 \times 10^3)^2} \\ &= \sqrt{10^6 + 7.023 \times 10^6} \\ &= \sqrt{8.023 \times 10^6} \\ &= 2.83 \times 10^3 \\ &= 2{,}830 \text{ ohms} \end{aligned}$$

Example 2: A 300-volt, variable-frequency ac generator is connected in series with an inductive reactance of 300 ohms, a capacitive reactance of 100 ohms, and a resistance of 100 ohms. What is the impedance of the circuit?

$$\begin{aligned} Z &= \sqrt{R^2 + (X_L - X_C)^2} \\ &= \sqrt{(100)^2 + (300 - 100)^2} \\ &= \sqrt{(100)^2 + (200)^2} \\ &= \sqrt{(10^2)^2 + (2 \times 10^2)^2} \\ &= \sqrt{10^4 + 4 \times 10^4} \\ &= \sqrt{5 \times 10^4} \\ &= 2.236 \times 10^2 \\ &= 223.6 \text{ ohms} \end{aligned}$$

199. Solving Ac Circuits Having Resistance and Inductance

a. Series Circuits. The following examples illustrate the method of solving series ac circuits having resistance and inductance (called series RL circuits) by using the principles described in paragraphs 193 through 198.

Example 1: An ac circuit with a resistance of 1,000 ohms and an inductance of 5 henrys is connected in series with a generator (fig. 71). The voltage drop across the resistance is 51.5 volts, and the voltage drop across the inductance is 97 volts. Find the applied voltage in the circuit. If the impedance of the circuit is 2,132 ohms, what is the phase angle by which the current lags the applied voltage?

Step 1. The vector diagram for this circuit is shown in figure 77. In an ac series circuit, the same current flows through all parts of the circuit—in this case, 0.051 ampere. Draw the vector E_R to represent the voltage drop across the resistance. Draw the vector E_L to represent the voltage drop across the inductance.

Step 2. The vector sum of these voltage drops is equal to the applied voltage. Adding the horizontal and vertical voltage drops, respectively:

$$E_L + E_R = 0 + 51.5 = 51.5$$
$$E_L + E_R = 97 + 0 = 97$$

Step 3. Find the applied voltage as follows:

$$\begin{aligned} E^2 &= E_L + E_R^2 \\ &= (97)^2 + (51.5)^2 \\ &= 9409 + 2652.25 \\ &= 12061.25 \\ E &= \sqrt{12061.25} \\ &= 109.8 \text{ or approx } 110 \text{ volts} \end{aligned}$$

Step 4. Find the phase angle by which the current lags the applied voltage.

$$\begin{aligned} \cos A &= \frac{R}{Z} \text{ (for series circuit)} \\ &= \frac{1000}{2132} \\ &= 0.46904 \\ A &= 62^\circ\ 1'\ 19'' \end{aligned}$$

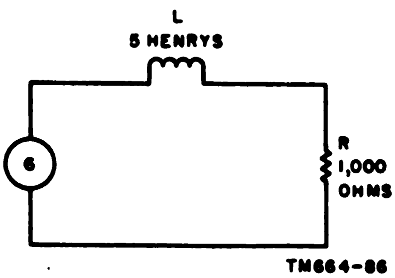


Figure 71. An ac series circuit containing inductance and resistance.

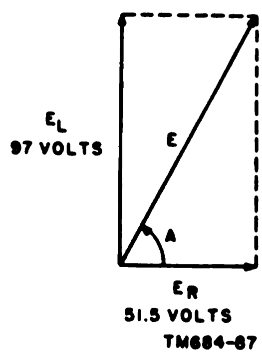


Figure 72. Ac series circuit containing inductance and resistance, vector diagram.

Step 5. Therefore, the current lags the applied voltage by a phase angle of 62° 1′ 19″.

Example 2: A 110-volt, 60-cycle ac generator is connected in a series circuit to a load consisting of an inductance of 3 henrys and a resistance of 10,000 ohms (A, fig. 73).

Step 1. Find the inductive reactance of the circuit.

$$\begin{aligned} X_L &= 2\pi fL \\ &= 2 \times 3.14 \times 60 \times 3 \\ &= 6.28 \times 180 \\ &= 1130.4 \\ &= 1{,}130 \text{ ohms (approx)} \end{aligned}$$

Step 2. Find the impedance of the circuit.

$$\begin{aligned} Z &= \sqrt{R^2 + X_L^2} \\ &= \sqrt{(10{,}000)^2 + (1130)^2} \\ &= \sqrt{100{,}000{,}000 + 1{,}276{,}900} \\ &= \sqrt{101{,}276{,}900} \\ &= 10{,}063.64 \\ &= 10{,}064 \text{ ohms (approx)} \end{aligned}$$

Step 3. Find the effective current in the circuit. (The effective value is the equivalent heating value of an alternating current as compared to a direct current. It is also called the root-mean-square (rms) value.)

$$\begin{aligned} I &= \frac{E}{Z} \\ &= \frac{110}{10{,}065} \\ &= 0.0109 \text{ ampere} \end{aligned}$$

Step 4. In a series circuit, the same current flows through all parts of the circuit. Therefore, the current through both the inductance and the resistance is 0.0109 ampere.

Step 5. Find the voltage drop across the inductance.

$$\begin{aligned} E_L &= IX_L \\ &= 0.0109 \times 1130 \\ &= 12.317 \\ &= 12 \text{ volts (approx)} \end{aligned}$$

Step 6. Find the voltage drop across the resistance.

$$\begin{aligned} E_R &= IR \\ &= 0.0109 \times 10{,}000 \\ &= 109 \text{ volts} \end{aligned}$$

Step 7. Find the total voltage in the circuit. In an ac series circuit, voltage drops are added vectorially (B, fig. 73).

$$\begin{aligned} E_t^2 &= E_L^2 + E_R^2 \\ &= (12)^2 + (109)^2 \\ &= 144 + 11{,}881 \\ &= 12{,}025 \\ E_t &= \sqrt{12{,}025} \\ &= 109.6 \\ &= 110 \text{ volts (approx)} \end{aligned}$$

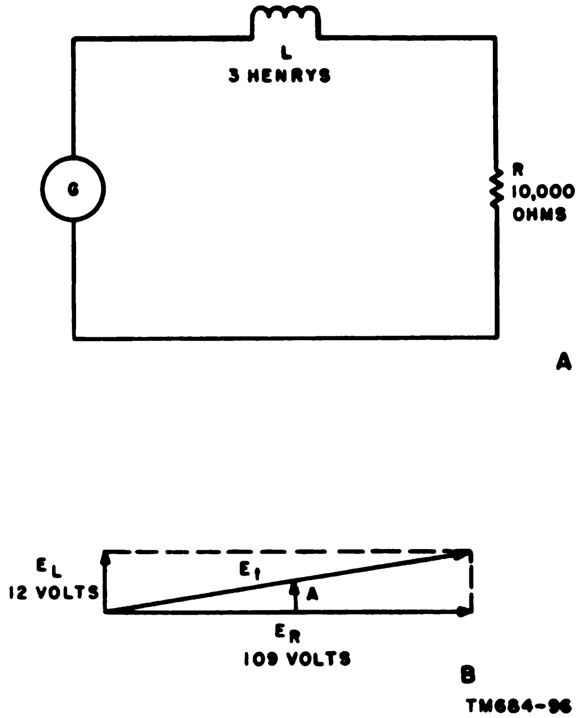


Figure 73. Ac series circuit having inductance and resistance, schematic and vector diagrams.

Step 8. Find the phase angle by which the current lags the applied voltage.

$$\cos A = \frac{R}{Z}$$

$$= \frac{10{,}000}{10{,}065}$$

$$= 0.99354$$

$$A = 6° \; 31'$$

b. Parallel Circuits. The following examples illustrate the method of solving parallel RL circuits by using the principles described in paragraphs 193 through 198.

Example 1: An ac circuit has an inductance and resistance connected in parallel (fig. 74). The current flowing through the inductance is 0.0584 ampere, and the current flowing through the resistance is 0.11 ampere. What is the total current in the circuit? If the impedance of the circuit is 884 ohms, what is the phase angle by which the line current lags the applied voltage?

Step 1. The vector diagram for this circuit is shown in figure 75. In a parallel circuit the voltage drop across each inductance or resistance is the same—in this circuit, 110 volts. Draw the vector I_R to represent the current through the resistor. Draw the vector I_L to represent the current through the inductance.

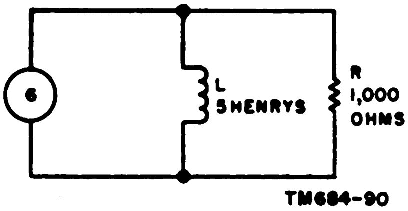


Figure 74. An ac parallel circuit containing inductance and resistance.

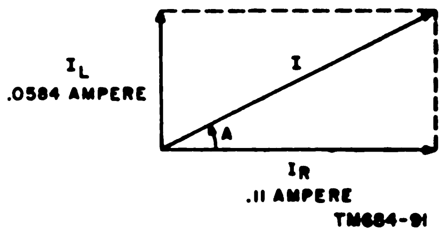


Figure 75. Ac parallel circuit containing inductance and resistance, vector diagram.

Step 2. The horizontal and vertical currents, respectively are:

$$I_R = 0.11 \text{ ampere}$$
$$I_L = 0.0584 \text{ ampere}$$

Step 3. Find the total current as follows:

$$I^2 = I_L^2 + I_R^2$$
$$= (0.0584)^2 + (0.11)^2$$
$$= 0.0034 + .0121$$
$$= 0.0155$$
$$I = \sqrt{0.0155}$$
$$= 0.1245 \text{ ampere}$$

Step 4. Find the phase angle by which the line current lags the applied voltage.

$$\cos A = \frac{Z}{R} \text{ (for parallel circuit)}$$
$$= \frac{884}{1{,}000}$$
$$= 0.88400$$
$$A = 27^\circ\ 52'\ 43''$$

Step 5. Thus, the line current lags the applied voltage by a phase angle of 27° 52′ 43″.

Example 2: A 110-volt, 60-cycle ac generator is connected in a parallel circuit to a load consisting of an inductance of 3 henrys and a resistance of 10,000 ohms (A, fig. 76).

Step 1. Find the inductive reactance of the circuit.

$$X_L = 2\pi fL$$
$$= 2 \times 3.14 \times 60 \times 3$$
$$= 6.28 \times 180$$
$$= 1130.4$$
$$= 1130 \text{ ohms (approx)}$$

Step 2. Find the impedance of the circuit.

$$Z = \frac{RX_L}{\sqrt{R^2 + X_L^2}}$$
$$= \frac{10{,}000 \times 1130}{\sqrt{(10{,}000)^2 + (1130)^2}}$$
$$= \frac{10^4 \times 1.13 \times 10^3}{\sqrt{(10^4)^2 + (1.13 \times 10^3)^2}}$$
$$= \frac{1.13 \times 10^7}{\sqrt{10^8 + 1.277 \times 10^6}}$$

$$= \frac{1.13 \times 10^7}{\sqrt{100 \times 10^6 + 1.277 \times 10^6}}$$
$$= \frac{1.13 \times 10^7}{\sqrt{101.277 \times 10^6}}$$
$$= \frac{1.13 \times 10^7}{10.07 \times 10^3}$$
$$= .1123 \times 10^4$$
$$= 1123 \text{ ohms (approx)}$$

Step 3. Find the line current in the circuit.

$$I = \frac{E}{Z}$$
$$= \frac{110}{1123}$$
$$= 0.09795 \text{ ampere}$$

Step 4. Find the current flowing through the inductance.

$$I_L = \frac{E}{X_L}$$
$$= \frac{110}{1130}$$
$$= .09734$$
$$= 0.0973 \text{ ampere (approx)}$$

Step 5. Find the current flowing through the resistance.

$$I_R = \frac{E}{R}$$
$$= \frac{110}{10{,}000}$$
$$= 0.011 \text{ ampere}$$

Step 6. Find the total current in the circuit. In an ac parallel circuit, the currents through the separate parts of the circuit are added vectorially (B, fig. 76).

$$I_t^2 = I_L^2 + I_R^2$$
$$= (0.097)^2 + (0.011)^2$$
$$= (9.7 \times 10^{-2})^2 + (1.1 \times 10^{-2})^2$$
$$= 94.09 \times 10^{-4} + 1.21 \times 10^{-4}$$
$$= 95.3 \times 10^{-4}$$
$$I_t = \sqrt{95.3 \times 10^{-4}}$$
$$= 9.8 \times 10^{-2}$$
$$= .098 \text{ ampere (approx)}$$

Step 7. In a parallel circuit, the voltage drop across one element would be the same as the voltage drop across another element in parallel with it. Thus, the voltage drop across both the inductance and the resistance is 110 volts.

Step 8. Find the phase angle by which the line current lags the applied voltage.

$$\cos A = \frac{Z}{R}$$
$$= \frac{1123}{10{,}000}$$
$$= 0.11230$$
$$A = 83^\circ\, 33'\, 52''$$

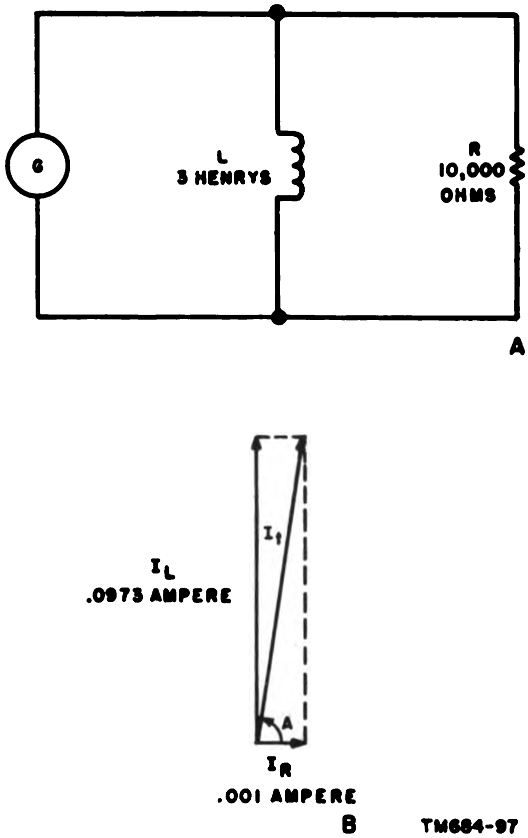


Figure 76. Ac parallel circuit having inductance and resistance, schematic and vector diagrams.

200. Solving Ac Circuits Having Resistance and Capacitance

a. Series Circuits. The following examples illustrate the method of solving series ac circuits having resistance and capacitance (called series RC circuits) by using the principles described in paragraphs 193 through 198.

Example 1: An ac generator in a series circuit is connected to a load consisting of a capacitance and a resistance (fig. 77). The voltage drop across the capacitance is 103 volts, and the voltage drop across the resistance is 39 volts. What is the applied voltage in the circuit? If the impedance of the circuit is 2,840 ohms, what is the phase angle by which the current leads the applied voltage?

Step 1. The vector diagram for this circuit is shown in figure 78. In a series circuit, the same current flows through all parts of the circuit—in this case, 0.039 ampere. Draw the vector E_R to represent the voltage drop across the resistance. Draw the vector E_C to represent the voltage drop across the capacitance.

Step 2. The vector sum of these voltage drops is equal to the applied voltage. Adding the horizontal and vertical voltage drops, respectively:

$$E_C + E_R = 0 + 39 = 39$$
$$E_C + E_R = 103 + 0 = 103$$

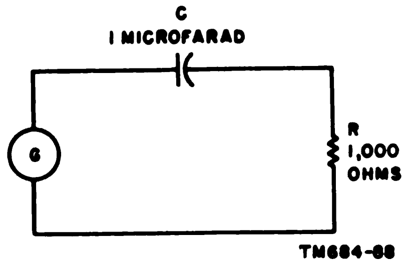


Figure 77. An ac series circuit containing capacitance and resistance.

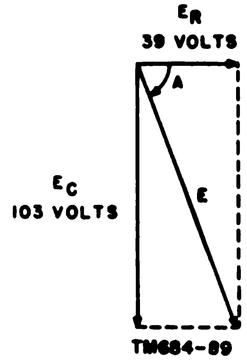


Figure 78. Ac series circuit containing capacitance and resistance, vector diagram.

Step 3. Find the applied voltage as follows:

$$\begin{aligned} E^2 &= E_C^2 + E_R^2 \\ &= (103)^2 + (39)^2 \\ &= (1.03 \times 10^2)^2 + (3.9 \times 10)^2 \\ &= 1.061 \times 10^4 + 15.2 \times 10^2 \\ &= 106.1 \times 10^2 + 15.2 \times 10^2 \\ &= 121.3 \times 10^2 \\ E &= \sqrt{121.3 \times 10^2} \\ &= 11.01 \times 10 \\ &= 110.1 \text{ volts} \end{aligned}$$

Step 4. Find the phase angle by which the current leads the applied voltage.

$$\begin{aligned} \cos A &= \frac{R}{Z} \\ &= \frac{1000}{2840} \\ &= 0.35211 \\ &= 69^\circ\, 24' \end{aligned}$$

Step 5. Thus, the current leads the applied voltage by a phase angle of $69^\circ\, 24'$.

Example 2: A 110-volt, 60-cycle ac generator is connected in a series circuit to a load consisting of a 2-microfarad capacitor and a 10,000-ohm resistor (A, fig. 79).

Step 1. Find the capacitive reactance of the circuit.

$$\begin{aligned} X_C &= \frac{1}{2\pi fC} \\ &= \frac{1}{2 \times 3.14 \times 60 \times 2 \times 10^{-6}} \\ &= \frac{1}{753.6 \times 10^{-6}} \\ &= \frac{1}{7.536 \times 10^{-4}} \\ &= \frac{10^4}{7.536} \\ &= \frac{10,000}{7.536} \\ &= 1,327 \text{ ohms (approx)} \end{aligned}$$

Step 2. Find the impedance of the circuit.

$$\begin{aligned} Z &= \sqrt{R^2 + X_C^2} \\ &= \sqrt{(10{,}000)^2 + (1327)^2} \\ &= \sqrt{(10^4)^2 + (1.327 \times 10^3)^2} \\ &= \sqrt{10^8 + 1.761 \times 10^6} \\ &= \sqrt{100 \times 10^6 + 1.761 \times 10^6} \\ &= \sqrt{101.761 \times 10^6} \\ &= 10.088 \times 10^3 \\ &= 10{,}088 \text{ ohms (approx)} \end{aligned}$$

Step 3. Find the current in the circuit.

$$\begin{aligned} I &= \frac{E}{Z} \\ &= \frac{110}{10{,}088} \\ &= 0.0109 \text{ ampere (approx)} \end{aligned}$$

Step 4. In a series circuit, the same current flows through all parts of the circuit; therefore, the current through both the capacitance and the resistance is 0.0109 ampere.

Step 5. Find the voltage drop across the capacitance.

$$\begin{aligned} E_C &= IX_C \\ &= 0.0109 \times 1327 \\ &= 14.46 \\ &= 14 \text{ volts} \end{aligned}$$

Step 6. Find the voltage drop across the resistance.

$$\begin{aligned} E_R &= IR \\ &= 0.0109 \times 10{,}000 \\ &= 109 \text{ volts} \end{aligned}$$

Step 7. Find the total voltage in the circuit (B, fig. 79).

$$\begin{aligned} E_t^2 &= E_R^2 + E_C^2 \\ &= (109)^2 + (14)^2 \\ &= (1.09 \times 10^2)^2 + (1.4 \times 10)^2 \\ &= 1.1881 \times 10^4 + 1.96 \times 10^2 \\ &= 118.81 \times 10^2 + 1.96 \times 10^2 \\ &= 120.77 \times 10^2 \\ E &= \sqrt{120.77 \times 10^2} \\ &= 10.99 \times 10 \\ &= 109.9 \text{ or } 110 \text{ volts} \end{aligned}$$

Step 8. Find the phase angle by which the current leads the applied voltage.

$$\begin{aligned} \cos A &= \frac{R}{Z} \\ &= \frac{10{,}000}{10{,}088} \\ &= 0.991178 \\ &= 0.99118 \\ A &= 7^\circ\ 37' \end{aligned}$$

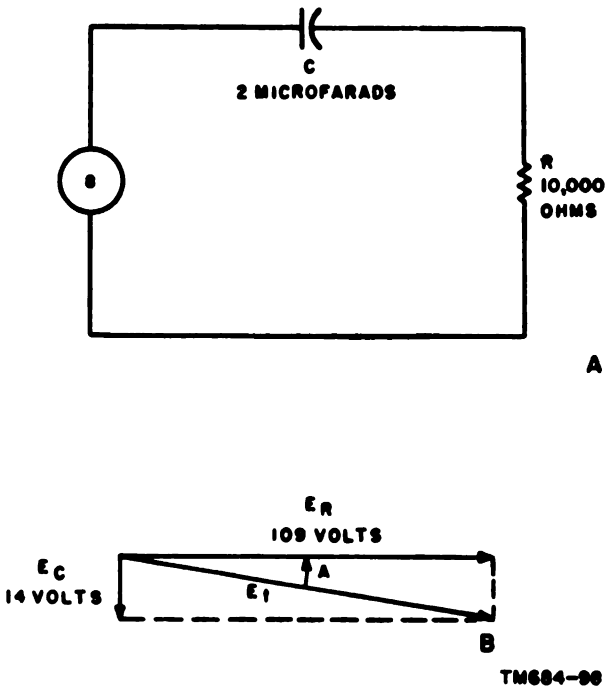


Figure 79. Ac series circuit having capacitance and resistance, schematic and vector diagrams.

b. Parallel Circuits. The following examples illustrate the method of solving parallel *RC* circuits by using the principles described in paragraphs 193 through 198.

Example 1: An ac circuit has a capacitance and resistance connected in parallel (fig. 80). The current flowing through the capacitance is 0.0415 ampere, and the current flowing through the resistance is 0.11 ampere. What is the total current in the circuit? If the impedance of the circuit is 938 ohms, what is the phase angle by which the current leads the applied voltage?

Step 1. The vector diagram for this circuit is shown in figure 81. In a parallel circuit, the voltage drop across each capacitance or resistance is the same—in this case, 110 volts. Draw the vector I_R to represent the current through the resistor. Draw the vector I_C to represent the current through the capacitance.

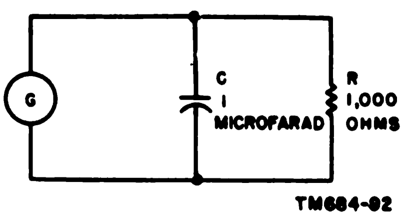


Figure 80. An ac parallel circuit containing capacitance and resistance.

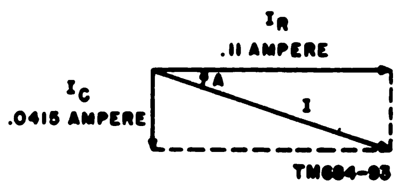


Figure 81. Ac parallel circuit containing capacitance and resistance, vector diagram.

Step 2. The vector sum of the currents through the separate parts of the circuit will be equal to the total current. Adding the horizontal and vertical currents, respectively:

$$I_C + I_R = 0 + .11 = .11$$

$$I_C + I_R = .0415 + 0 = .0415$$

Step 3. Find the total current as follows:

$$\begin{aligned} I^2 &= I_C^2 + I_R^2 \\ &= (.0145)^2 + (.11)^2 \\ &= (1.45 \times 10^{-2})^2 + (11 \times 10^{-2})^2 \\ &= 2.1 \times 10^{-4} + 121 \times 10^{-4} \\ &= 123.1 \times 10^{-4} \\ I &= \sqrt{123.1 \times 10^{-4}} \\ &= 11.1 \times 10^{-2} \\ &= .111 \text{ ampere} \end{aligned}$$

Step 4. Find the phase angle by which the current leads the applied voltage.

$$\begin{aligned} \cos A &= \frac{Z}{R} \\ &= \frac{938}{1{,}000} \\ &= .93800 \\ A &= 20^\circ\ 17'\ 6'' \end{aligned}$$

Step 5. Thus, the current leads the applied voltage by a phase angle of 20° 17′ 6″.

Example 2: A 110-volt, 60-cycle ac generator is connected to a load consisting of a 2-microfarad capacitance and a 10,000-ohm resistance in parallel (A, fig. 82).

Step 1. Find the capacitance reactance of the circuit.

$$\begin{aligned} X_C &= \frac{1}{2\pi fC} \\ &= \frac{1}{2 \times 3.14 \times 60 \times 2 \times 10^{-6}} \\ &= \frac{1}{7.536 \times 10^{-4}} \\ &= \frac{10^4}{7.536} \\ &= \frac{10{,}000}{7.536} \\ &= 1{,}327 \text{ ohms} \end{aligned}$$

Step 2. Find the impedance of the circuit.

$$\begin{aligned} Z &= \frac{RX_C}{\sqrt{R^2 + X_C^2}} \\ &= \frac{10{,}000 \times 1327}{\sqrt{(10{,}000)^2 + (1327)^2}} \\ &= \frac{10^4 \times 1.327 \times 10^3}{\sqrt{(10^4)^2 + (1.327 \times 10^3)^2}} \\ &= \frac{1.327 \times 10^7}{\sqrt{10^8 + 1.76 \times 10^6}} \\ &= \frac{1.327 \times 10^7}{\sqrt{100 \times 10^6 + 1.76 \times 10^6}} \end{aligned}$$

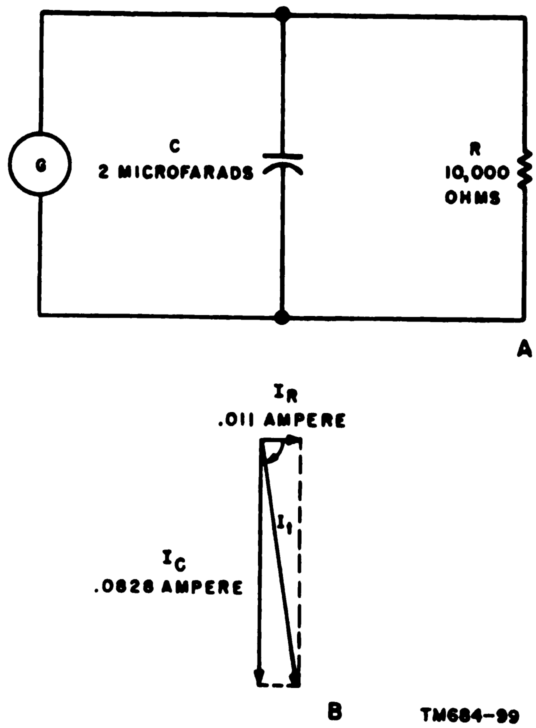


Figure 82. Ac parallel circuit having resistance and capacitance, schematic and vector diagrams.

$$= \frac{1.327 \times 10^7}{\sqrt{101.76 \times 10^6}}$$

$$= \frac{1.327 \times 10^7}{10.088 \times 10^3}$$

$$= .1315 \times 10^4$$

$= .1315$ ohms (approx)

Step 3. Find the current flowing through the capacitance.

$$I_C = \frac{E}{X_C}$$

$$= \frac{110}{1327}$$

$$= 0.08289$$

$= 0.0829$ ampere

Step 4. Find the current flowing through the resistance.

$$I_R = \frac{E}{R}$$

$$= \frac{110}{10,000}$$

$= 0.011$ ampere

Step 5. Find the total current in the circuit.

$$I_t^2 = I_R^2 + I_C^2$$

$$= (.011)^2 + (.0829)^2$$

$$= (1.1 \times 10^{-2})^2 + (8.29 \times 10^{-2})^2$$

$$= 1.21 \times 10^{-4} + 68.72 \times 10^{-4}$$

$$= 69.93 \times 10^{-4}$$

$$I_t = \sqrt{69.93 \times 10^{-4}}$$
$$= 8.36 \times 10^{-2}$$
$$= .0836 \text{ ampere (approx)}$$

Step 6. In a parallel circuit, the voltage drop across each capacitance or resistance in parallel is the same. Thus, the voltage drop across both the capacitance and the resistance is 110 volts.

Step 7. Find the phase angle by which the current leads the applied voltage.

$$\cos A = \frac{Z}{R}$$
$$= \frac{1315}{10{,}000}$$
$$= 0.13150$$
$$A = 82° \, 26' \, 37''$$

201. Solving Ac Circuits Having Resistance, Inductance, and Capacitance

a. Series Circuits. The following examples illustrate the method of solving series ac circuits having resistnace, inductance, and capacitance (called series RLC circuits) by using the principles described in paragraphs 193 through 198.

Example 1: A 300-volt, 60-cycle ac generator is connected in series with a 6-ohm resistance, an 8-ohm inductive reactance, and a 16-ohm capacitive reactance (fig. 83). Find (1) the resultant reactive voltage, (2) the current flowing in the circuit, and (3) the voltage drops across the resistance, the inductance, and the capacitance. (4) Check the solution by vectorially adding E_L, E_C, and E_R. The result should equal the applied voltage. (5) Find the phase angle by which the current leads or lags the applied voltage.

Step 1. The vector diagram for this circuit is shown in figure 84. Since E_C and E_L are 180° out of phase, their vector sum

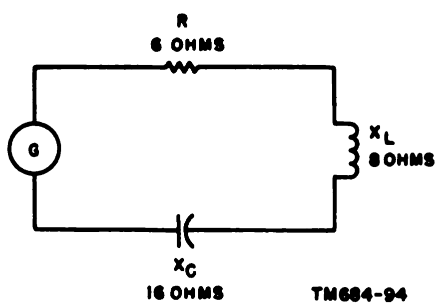


Figure 83. An ac series circuit containing resistance, inductive reactance, and capacitance reactance.

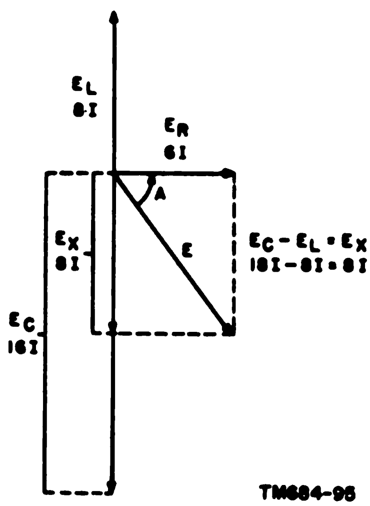


Figure 84. Ac series circuit containing resistance, inductive reactance, and capacitive reactance vector diagram.

is the difference between the two. E_C is greater than E_L; thus, the resultant reactive voltage, E_X, is $16I - 8I = 8I$.

Step 2. Find the current flowing in the circuit.

$$
\begin{aligned}
E^2 &= E_R^2 + EX^2 \\
E^2 &= (IR)^2 + (IX)^2 \\
(300)^2 &= (6I)^2 + (8I)^2 \\
(3 \times 10^2)^2 &= (6I)^2 + (8I)^2 \\
9 \times 10^4 &= 36I^2 + 64I^2 \\
9 \times 10^4 &= 100I^2 \\
9 \times 10^4 &= 10^2I^2 \\
\sqrt{9 \times 10^4} &= \sqrt{10^2I^2} \\
3 \times 10^2 &= 10I \\
300 &= 10I \\
I &= 30 \text{ amperes}
\end{aligned}
$$

Step 3. Find the voltage drop across the resistance.

$$
\begin{aligned}
E_R &= IR \\
&= 30 \times 6 \\
&= 180 \text{ volts}
\end{aligned}
$$

Step 4. Find the voltage drop across the inductance.

$$
\begin{aligned}
E_L &= IX_L \\
&= 30 \times 8 \\
&= 240 \text{ volts}
\end{aligned}
$$

Step 5. Find the voltage drop across the capacitance.

$$
\begin{aligned}
E_C &= IX_C \\
&= 30 \times 16 \\
&= 480 \text{ volts}
\end{aligned}
$$

Step 6. Find the resultant reactive voltage.

$$
\begin{aligned}
E_X &= E_C - E_L \\
&= 480 - 240 \\
&= 240 \text{ volts}
\end{aligned}
$$

Step 7. Vectorially add the voltages in the circuit. The result should equal the applied voltage.

$$
\begin{aligned}
E^2 &= E_R^2 + E_X^2 \\
&= (180)^2 + (240)^2 \\
&= (1.8 \times 10^2)^2 + (2.4 \times 10^2)^2 \\
&= 3.24 \times 10^4 + 5.76 \times 10^4 \\
&= 9 \times 10^4 \\
E &= \sqrt{9 \times 10^4} \\
&= 3 \times 10^2 \\
&= 300 \text{ volts}
\end{aligned}
$$

Step 8. Find the phase angle by which the current leads or lags the applied voltage in the circuit.

$$
\begin{aligned}
\tan A &= \frac{X_L - X_C}{R} \\
&= \frac{8}{6} \\
&= 1.33333 \\
A &= 53^\circ\ 7'\ 48''.
\end{aligned}
$$

Step 9. The circuit is predominantly capacitive; therefore, the current *leads* the applied voltage by a phase angle of 53° 7′ 48″.

Example 2: A 60-cycle ac generator is connected in series with a 10-ohm resistance, a 12-ohm inductive reactance, and a 20-ohm capacitive reactance (A, fig. 85). The current flowing through the circuit is 19 amperes. (1) Find the voltage drop across each circuit element. (2) Find the total voltage. (3) Find the phase angle between the current and the applied voltage.

Step 1. Find the voltage drop across the resistance.

$$E_R = IR$$
$$= 19 \times 10$$
$$= 190 \text{ volts}$$

Step 2. Find the voltage drop across the inductance.

$$E_L = IX_L$$
$$= 19 \times 12$$
$$= 228 \text{ volts}$$

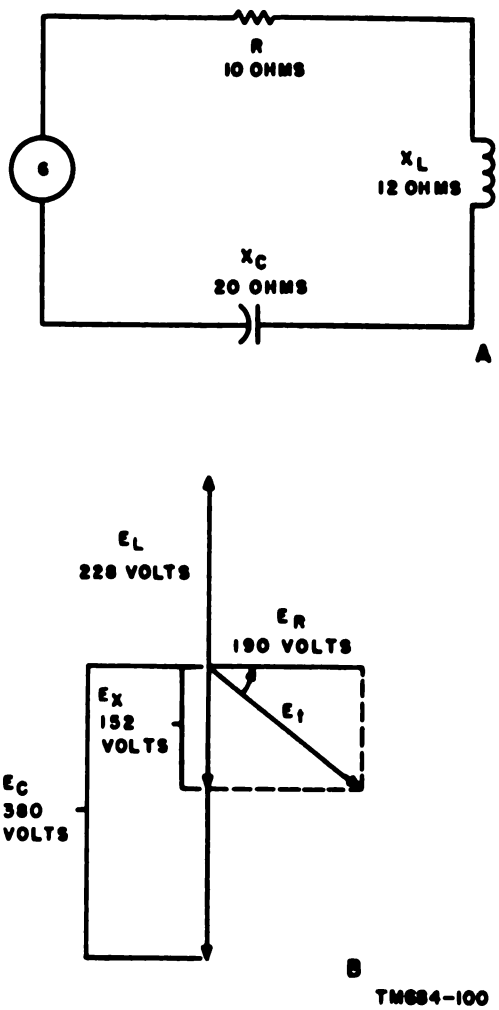


Figure 85. An ac series circuit having inductance, capacitance, and resistance, schematic and vector diagrams.

Step 3. Find the voltage drop across the capacitance.

$$E_C = IX_C$$
$$= 19 \times 20$$
$$= 380 \text{ volts}$$

Step 4. Find the resultant reactive voltage.

$$E_X = E_C - E_L$$
$$= 380 - 228$$
$$= 152 \text{ volts}$$

Step 5. Find the total voltage in the circuit.

$$E^2 = E_R^2 + E_X^2$$
$$= (190)^2 + (152)^2$$
$$= (1.9 \times 10^2)^2 + (1.52 \times 10^2)^2$$
$$= 3.61 \times 10^4 + 2.31 \times 10^4$$
$$= 5.92 \times 10^4$$
$$E = \sqrt{5.92 \times 10^4}$$
$$= 2.43 \times 10^2$$
$$= 243 \text{ volts}$$

Step 6. Find the phase angle by which the current leads the applied voltage in the circuit. Since the capacitive reactance is greater and cancels the inductive reactance, the circuit is capacitive and the current leads the applied voltage by the phase angle A.

$$\tan A = \frac{X_C - X_L}{R}$$
$$= \frac{20 - 12}{10}$$
$$= \frac{8}{10}$$
$$= .80000$$
$$A = 38° \; 39' \; 35''$$

b. Parallel Circuits. The following example illustrates the method of solving parallel ac circuits having resistance, inductance, and capacitance (called parallel RLC circuits) by using the principles described in paragraphs 193 through 198.

Example: A parallel circuit has a 300-volt input, a 150-ohm resistance, a 125-ohm inductive reactance, and a 100-ohm capacitive reactance (A, fig. 86).

Step 1. Since this is a parallel circuit, the same voltage is impressed across the inductance, the resistance, and the capacitance. Thus, the voltage across each of them is 300 volts.

Step 2. Find the current flowing through the resistor.

$$I_R = \frac{E}{R}$$
$$= \frac{300}{150}$$
$$= 2 \text{ amperes}$$

Step 3. Find the current flowing through the inductance.

$$I_L = \frac{E}{X_L}$$
$$= \frac{300}{125}$$
$$= 2.4 \text{ amperes}$$

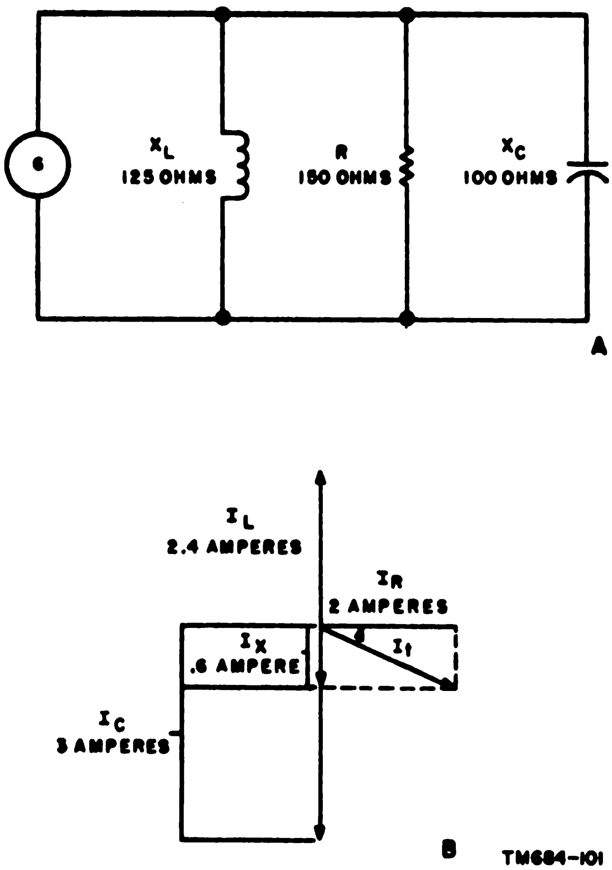


Figure 86. Ac parallel circuit having inductance, capacitance, and resistance, schematic and vector diagrams.

Step 4. **Find the current flowing through the capacitor.**

$$I_C = \frac{E}{X_C}$$
$$= \frac{300}{100}$$
$$= 3 \text{ amperes}$$

Step 5. **Find the total current in the circuit (B, fig. 86).**

$$I_X = I_C - I_L$$
$$= 3 - 2.4$$
$$= .6 \text{ ampere}$$
$$I_t^2 = I_R^2 + I_X^2$$
$$= (2)^2 + (.6)^2$$
$$= 4 + .36$$
$$= 4.36$$
$$I_t = \sqrt{4.36}$$
$$= 2.0889$$
$$= 2.089 \text{ amperes}$$

Step 6. Find the impedance of the circuit.

$$
\begin{aligned}
Z &= \frac{E}{I_t} \\
&= \frac{300}{2.089} \\
&= 143.6 \\
&= 144 \text{ ohms (approx)}
\end{aligned}
$$

Step 7. Find the phase angle by which the current leads the applied voltage. Since this is a parallel circuit in which the inductive reactance is greater than the capacitive reactance, the circuit is capacitive and the current leads the applied voltage.

$$
\begin{aligned}
\cos A &= \frac{Z}{R} \\
&= \frac{144}{150} \\
&= .96000 \\
A &= 16^\circ\, 15'\, 38''
\end{aligned}
$$

202. Resonance

In a series or parallel ac circuit containing inductance and capacitance, a condition known as *resonance* exists when the inductive reactance equals the capacitive reactance. This condition occurs at a specfiic frequency called the *resonant frequency.* A formula for finding the resonant frequency is derived by equating the formulas for inductive reactance and capacitive reactance, as follows:

$$
\begin{aligned}
X_L &= X_C \\
2\pi f_r L &= \frac{1}{2\pi f_r C} \\
4\pi^2 f_r^2 LC &= 1 \\
f_r^2 &= \frac{1}{4\pi^2 LC} \\
f_r &= \frac{1}{2\pi\sqrt{LC}}
\end{aligned}
$$

where f_r is the resonant frequency.

Example: Find the resonant frequency of a circuit containing a 4-millihenry inductance and a 40-micromicrofarad capacitor in series with a variable frequency ac source.

$$
\begin{aligned}
f &= \frac{1}{2\pi\sqrt{LC}} \\
&= \frac{1}{6.28\sqrt{4 \times 10^{-3} \times 4 \times 10^{-11}}} \\
&= \frac{1}{6.28 \times 4 \times 10^{-7}} \\
&= \frac{10^7}{25.12} \\
&= \frac{10,000,000}{25.12} \\
&= 398,000 \text{ cps or } 398 \text{ kilocycles (kc).}
\end{aligned}
$$

203. Ac Power

a. In an ac circuit containing both resistance and reactance the only power actually dissipated is the power absorbed by the resistance of the circuit (*b* below). However, if the circuit contained reactance only, large amounts of power would still *appear* to be consumed because of the phase difference between voltage and current. Consequently, in either case an ac generator supplying power to the circuit would receive less power from the circuit than it delivers to the circuit. The power which the generator delivers to the circuit is called the *apparent* power and is equal to the product of the effective value of the voltage ($E_{max}/\sqrt{2}$) and the effective value of the current ($I_{max}/\sqrt{2}$). Therefore,

$$P \text{ (apparent power)} = \frac{E_{max}}{\sqrt{2}} \cdot \frac{I_{max}}{\sqrt{2}} = \frac{E_{max}\, I_{max}}{2}$$

b. Apparent power is different from the actual power consumed by the load, which is called the average or true power and is the energy absorbed by the resistance of the circuit. The average or true power is expressed by the formula $P = EI \cos \theta$, where

E = effective value of the voltage across the circuit

I = effective value of the current in the circuit

θ = phase angle between current and voltage

c. Apparent power may also be expressed by the following formulas:

$$P = EI$$

$$P = I^2Z$$

$$P = \frac{E^2}{Z}$$

d. In a purely resistive circuit, average or true ac power also may be expressed by Joule's law ($P = I^2R$) as in the dc case (par. 191).

e. The following examples illustrate some of the above principles.

Example 1: Find the power that an ac generator must deliver to a circuit if the peak voltage is 230 volts and the peak current is 5 amperes.

$$P = \frac{E_{max}\, I_{max}}{2} = \frac{230 \times 5}{2} = 575 \text{ watts.}$$

Example 2: Find the average power consumed in a circuit if the effective ac voltage is 115 volts, the effective current is 7 amperes, and the current leads the voltage by 60°.

$$P = EI \cos \theta = 115 \times 7 \times \cos 60° = 115 \times 7 \times .5 = 402.5 \text{ watts.}$$

204. Review Problems—Ac Electricity

a. An alternator is connected to a 520-volt, 60-cycle ac parallel circuit having a resistance of 96 ohms, an inductance of 249 millihenrys, and a capacity of 19.8 microfarads. (1) Find the inductive reactance of the circuit. (2) Find the capacitive reactance of the circuit. (3) Determine whether the current leads or lags the voltage. (4) Find the impedance of the circuit. (5) Determine the value of the current in the circuit.

b. Determine the inductive reactance of a coil if the ac in the circuit has a frequency of 60 cps, and the inductance of the coil is 0.025 henry.

c. A 110-volt, 25-cycle ac generator is connected in series with a 0.1-microfarad capacitance and a 2,000-ohm resistance. What is the capacitive reactance of the circuit?

d. What is the value of the reactance of a circuit if the impedance $Z = 100$ ohms and the resistance $R = 60$ ohms?

e. Find the resonant frequency of a series RLC circuit if the inductance is 0.478 millihenry and the capacitance is 256 micromicrofarads.

f. A series RLC circuit consists of 6 ohms resistance, 8 ohms inductive reactance, and 16 ohms capacitive reactance. (1) Find the current in the circuit. (2) Find the voltage drop across the resistance. (3) Find the voltage drop across the capacitance. (4) Find the voltage drop across the inductance.

g. A parallel RLC circuit has an input voltage of 300 volts, an inductive reactance of 75 ohms, a capacitive reactance of 50 ohms, and a resistance of 100 ohms. (1) Find the current through the resistance. (2) Find the current through the inductance. (3) Find the current through the capacitance. (4) Find the total impedance of the circuit. (5) Find the phase angle between the line or total current and the applied voltage. (6) Find the average power. (7) Find the apparent power.

CHAPTER 16

APPLICATIONS OF LOGARITHMS TO TRANSMISSION PROBLEMS

205. The Transmission Unit

When signal power is transmitted along a transmission line, there is a power loss or attenuation; if an amplifier is used in the circuit, there may be a power gain. This loss or gain of power, resulting in a decrease or increase in the intensity of the signal, is measured in terms of the decibel (db). The decibel is a measure of power *ratio* and is probably the most widely used unit in communications. The formula for measuring transmission loss or gain is:

$$\text{db} = 10 \log_{10} \frac{P_1}{P_2}$$

where $\frac{P_1}{P_2}$ is the ratio of the two powers being compared (par. 206).

206. Converting Power Ratio to Decibels

When converting a power ratio into its decibel expression, represent the larger power as P_1 and the smaller power as P_2, regardless of whether the larger power is the input or output. Thus, the power ratio will always be greater than 1, and its logarithm will be a positive number. Prefix a plus sign to the answer if the power change is a gain (the power output greater than the power input); prefix a minus sign if the power change is a loss.

Example 1: The input power to a transmission line is 10 milliwatts, and the output power is 2.46 milliwatts. Express the power change in db.

$$\begin{aligned}\text{db} &= 10 \log \frac{P_1}{P_2} \\ &= 10 \log \frac{10}{2.46} \\ &= 10 \times \log 4.07 \\ &= 10 \times .6096 \\ &= -6.096\end{aligned}$$

Thus, the loss of the transmission line is 6.096 db, since input is greater than output.

Example 2: A repeater amplifier has an input power of 2 milliwatts and an output power of 400 watts. Calculate the power change.

$$\begin{aligned}\text{db} &= 10 \log \frac{P_1}{P_2} \\ &= 10 \log \frac{400}{.002} \\ &= 10 \times \log 200{,}000 \\ &= 10 \times 5.3010 \\ &= +53.01\end{aligned}$$

The gain of the repeater amplifier is 53.01 db, since output is greater than input.

207. Converting Decibels to Power Ratio

To find the power ratio when the gain or loss is expressed in decibels, reverse the procedure given in paragraph 206. If the number of decibels is positive, the circuit has a power gain and the output power is greater than the input power. If the number of decibels is negative, the circuit has a power loss and the output power is less than the input power. Insert the power change in decibels in the formula given in paragraph 200 and divide by 10; then find the antilog of both sides of the equation (par. 118) to obtain the power ratio.

Example 1: A circuit is known to have a power change of +12 db. Find the power ratio.

$$\begin{aligned}\text{db} &= 10 \log \frac{P_1}{P_2} \\ 12 \text{ db} &= 10 \log \frac{P_1}{P_2} \\ 1.2 &= \log \frac{P_1}{P_2}\end{aligned}$$

Find the antilog of both sides of the equation.

$$\frac{P_1}{P_2} = 15.85 \text{ of } 15.9$$

Since the number of decibels is given as positive, the circuit has a gain and its output power is 15.9 times its input power.

Example 2: A certain wire transmission circuit has a power change of —25 db. Calculate the power ratio.

$$\text{db} = 10 \log \frac{P_1}{P_2}$$

$$25 \text{ db} = 10 \log \frac{P_1}{P_2}$$

$$2.5 = \log \frac{P_1}{P_2}$$

$$\frac{P_1}{P_2} = 316.2$$

Since the number of decibels has a minus sign, the circuit attenuates power. The output power is less than the input power by a ratio of 1 to 316.2.

208. Review Problems—Transmission Problems

a. A network has a loss of 16 decibels. What power ratio correspond to this loss?

b. The input to a powerline 50 miles long is 210 milliwatts. The power delivered at the end of the line is 40 microwatts. What is the attenuation in decibels per mile?

c. A power of 10 milliwatts is required to drive an audiofrequency (af) amplifier. The output of the amplifier is 120 milliwatts. What is the gain in decibels?

d. What is the ratio of the output power to the input power if there is a power gain of 14 decibels?

CHAPTER 17

MISCELLANEOUS ELECTRICAL PROBLEMS

209. Efficiency

Efficiency is the ratio of output to input and usually is expressed in percent (ch. 2). Generators, motors, and other electrical devices often are rated according to their efficiency. To express efficiency in percent, write the ratio of output to input as a fraction, convert to a decimal, and then convert the decimal to a percent (par. 4).

Example: What is the efficiency of a generator that has an output of 60 kilowatts (kw) and an input of 75 kilowatts?

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{60}{75} = 0.80 = 80\%$$

210. Percent Overload

Another application of percent is the overload rating of motors, generators, etc. In this application, the amount of power, that can be applied to or taken from an electrical device, above the rated output, is expressed as a percent of the rated output.

Example 1: What is the percent of overload capacity of a generator that has a rated output of 500 watts and can provide a maximum of 550 watts?

$$\text{Overload} = \text{maximum power} - \text{rated power} = 550 - 500 = 50 \text{ watts}$$

$$\text{Percent overload} = \frac{\text{Overload}}{\text{Rated power}} = \frac{50}{500} = 10\%$$

Example 2: Find the maximum output of a generator that is rated at 1,500 watts, and has a 10 percent overload capacity.

$0.10 \times 1{,}500 = 150$ watts

$1{,}500 + 150 = 1{,}650$ watts maximum output.

211. Tolerances

A tolerance is an allowance for variations from the standard or specified value. In the manufacture of resistors, for example, the resistance is permitted to be within a specified percentage of the standard value. This percentage is indicated in the color code of the resistors.

Example: Find the possible low and high values of a 20,000-ohm resistor with a tolerance of ± 5 percent.

$0.05 \times 20{,}000 = 1{,}000$ ohms

$20{,}000 + 1{,}000 = 21{,}000$ ohms (high value)

$20{,}000 - 1{,}000 = 19{,}000$ ohms (low value)

Therefore, since the tolerance is plus or minus 5%, the value of the resistor should be between 21,000 and 19,000 ohms.

212. Transformer Relationships

a. General. In a transformer, relationships exist between the currents, voltages, impedances, and number of turns of wire in the windings. These relationships are expressed by equations containing ratios involving these quantities.

b. Relationship Between Voltage and Number of Turns. This relationship is expressed by the following equation:

$$\frac{E_p}{N_p} = \frac{E_s}{N_s}$$

where E_p is the voltage across the primary winding, N_p is the number of turns on the primary winding, E_s is the voltage across the secondary winding, and N_s is the number of turns on the secondary winding (fig. 87). The equation may also be written:

$$\frac{E_p}{E_s} = \frac{N_p}{N_s}$$

$$E_pN_s = E_sN_p$$

$$E_s = \frac{E_pN_s}{N_p} \text{ or } E_p\left(\frac{N_s}{N_p}\right)$$

$$E_p = \frac{E_sN_p}{N_s} \text{ or } E_s\left(\frac{N_p}{N_s}\right)$$

The ratios N_p/N_s *and* N_s/N_p are called the *turns ratios* and may be expressed as a single factor.

Example: Find the voltage across the secondary winding of a transformer if the primary voltage is 100 volts and the turns ratio from primary to secondary is 1 to 4.

$$\frac{N_p}{N_s} = \frac{1}{4} \text{ or } \frac{N_s}{N_p} = 4$$

$$E_s = E_p\left(\frac{N_s}{N_p}\right)$$

$$E_s = 100\ (4)$$

$$= 400 \text{ volts}$$

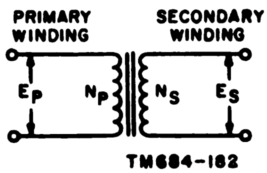


Figure 87. Simple Transformer.

c. Relationship Between Current and Number of Turns. This relationship is expressed by the following equation:

$$\frac{I_p}{I_s} = \frac{N_s}{N_p}$$

where I_p is the primary current, I_s is the secondary current, and N_p and N_s the number of turns on the primary and secondary as before. The equation may be written:

$$I_p\ N_p = I_sN_s$$

$$I_p = \frac{I_sN_s}{N_p} \text{ or, } I_s\left(\frac{N_s}{N_p}\right)$$

$$I_s = \frac{I_pN_p}{N_s} \text{ or, } I_p\left(\frac{N_p}{N_s}\right)$$

Example: Find the primary current in a transformer if the secondary current is 5 milliamperes and the turns ratio from primary to secondary is 20 to 1.

$$\frac{N_p}{N_s} = \frac{20}{1} \text{ or, } \frac{N_s}{N_p} = \frac{1}{20}$$

$$I_p = I_s\left(\frac{N_s}{N_p}\right)$$

$$= 5 \times 10^{-3}\left(\frac{1}{20}\right)$$

$$I_p = \frac{10^{-3}}{4}$$

$$= \frac{0.00100}{4}$$

$$= 0.00025 \text{ amperes, or, } 0.25 \text{ milliamperes}$$

d. Relationship Between Current and Voltage. By combining the relationships given in *b* and *c* above, a relationship can be derived between primary and secondary currents as follows:

$$\frac{E_p}{E_s} = \frac{N_p}{N_s} \text{ (from } b \text{ above)}$$

$$\frac{I_s}{I_p} = \frac{N_p}{N_s} \text{ (from } c \text{ above)}$$

Therefore, $\frac{E_p}{E_s} = \frac{I_s}{I_p}$ (because both are equal to $\frac{N_p}{N_s}$)

The equation may also be written:

$$E_p = \frac{E_s I_s}{I_p}$$

$$I_p = \frac{E_s I_s}{E_p}$$

$$E_s = \frac{E_p I_p}{I_s}$$

$$I_s = \frac{E_p I_p}{E_s}$$

$$E_p I_p = E_s I_s$$

Since voltage multiplied by current equals power, the last form of the equation states that the power absorbed by the primary winding is equal to the power delivered to the secondary winding. This is true in an ideal transformer which has no loss, and is essentially true in an actual transformer which has very little loss; efficiencies of 98 percent are common in actual transformers.

Example: Find the voltage across the secondary winding of a transformer if the primary voltage is 150 volts, the primary current is 5 amperes, and the secondary current is 25 amperes.

$$E_s = \frac{E_p I_p}{I_s}$$

$$= \frac{150 \times 5}{25}$$

$$= 30 \text{ volts}$$

e. Relationship Between Impedance and Number of Turns. This relationship also can be derived from the relationships given in *b* and *c* above by dividing one by the other, as follows:

$$\frac{E_p}{E_s} = \frac{N_p}{N_s} \text{ (from } b \text{ above)}$$

$$\frac{I_p}{I_s} = \frac{N_s}{N_p} \text{ (from } c \text{ above)}$$

$$\frac{\frac{E_p}{E_s}}{\frac{I_p}{I_s}} = \frac{\frac{N_p}{N_s}}{\frac{N_s}{N_p}} \text{ (dividing the first by the second)}$$

$$\frac{I_s}{I_p} \cdot \frac{E_p}{E_s} = \frac{N_p}{N_s} \quad \frac{N_p}{N_s}$$

$$\frac{E_p}{I_p} \cdot \frac{I_s}{E_s} = \frac{N_p^2}{N_s^2}$$

$$Z_p \cdot \frac{1}{Z_s} = \frac{N_p^2}{N_s^2} \text{ (substituting } Z \text{ for } \frac{E}{I}\text{)}$$

$$\frac{Z_p}{Z_s} = \frac{N_p^2}{N_s^2} \text{ or, } \frac{Z_p}{Z_s} = \left(\frac{N_p}{N_s}\right)^2$$

where Z_p is the impedance of the primary winding and Z_s is the impedance of the secondary winding in ohms. The equation may also be written:

$$Z_p = Z_s\left(\frac{N_p}{N_s}\right)^2$$

$$Z_s = Z_p\left(\frac{N_s}{N_p}\right)^2$$

Example: Find the impedance of the secondary winding of a transformer if the impedance of the primary winding is 200 ohms and the turns ratio from primary to secondary is 5 to 1.

$$\frac{N_p}{N_s} = \frac{5}{1} \text{ or, } \frac{N_s}{N_p} = \frac{1}{5}$$

$$Z_s = Z_p \left(\frac{N_s}{N_p}\right)^2$$

$$= 200 \left(\frac{1}{5}\right)^2$$

$$= 200 \times \frac{1}{25}$$

$$= 8 \text{ ohms}$$

213. Conductance

Conductance is a measure of the ease with which current flows in a circuit. It is given the symbol G and is equal to the reciprocal of resistance: $G = 1/R$. The unit of conductance is the *mho,* which is the word ohm spelled backwards.

Example: Find the conductance of a circuit consisting of a 4-ohm resistor in parallel with a 5-ohm resistor. In a parallel circuit, the reciprocal of the total resistance is equal to the sum of the reciprocals of the individual resistances:

$$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2},$$

$$\text{or } G = \frac{1}{R_1} + \frac{1}{R_2}$$

$$G = \frac{1}{4} + \frac{1}{5}$$

$$= .25 + .20$$

$$= .45 \text{ mhos}$$

214. Energy Stored in an Inductance

The amount of energy stored in an inductance is determined from the formula $P = \frac{LI^2}{2}$, where

L = inductance in henrys

I = current in amperes

P = energy in joules.

Example: Find the energy stored in a coil if the inductance is 7 millihenrys and the current is 3 milliamperes. Using scientific notation (par. 106), the energy in joules is:

$$P = \frac{LI^2}{2}$$

$$= \frac{7 \times 10^{-3} (3 \times 10^{-3})^2}{2}$$

$$= \frac{7 \times 10^{-3}\, 9 \times 10^{-6}}{2}$$

$$= \frac{63 \times 10^{-9}}{2}$$

$$= 31.5 \times 10^{-9} \text{ joules.}$$

215. Delta-Wye Transformations

a. A delta circuit consists of three resistors or other circuit components connected together to form the Greek letter *delta* (Δ). In a wye circuit, the resistors or other circuit components are connected together to form the letter Y. Figure 88 shows an example of each type of circuit constructed of resistors.

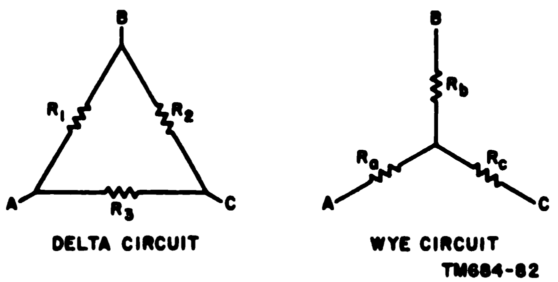


Figure 88. Delta and wye circuits.

b. If the resistances are known for a delta circuit, they can be found for an equivalent wye circuit from the following equations:

$$R_a = \frac{R_1 R_3}{R_1 + R_2 + R_3}$$

$$R_b = \frac{R_1 R_2}{R_1 + R_2 + R_3}$$

$$R_c = \frac{R_2 R_3}{R_1 + R_2 + R_3}$$

Example: **Find the equivalent resistances for a wye circuit if the resistances of a delta circuit are 10, 20, and 70 ohms.**

Let $R_1 = 10$ ohms, $R_2 = 20$ ohms, and $R_3 = 70$ ohms.

$$R_a = \frac{R_1R_3}{R_1 + R_2 + R_3}$$
$$= \frac{10 \times 70}{10 + 20 + 70}$$
$$= \frac{700}{100}$$
$$= 7 \text{ ohms}$$
$$R_b = \frac{R_1R_2}{R_1 + R_2 + R_3}$$
$$= \frac{10 \times 20}{100}$$
$$= 2 \text{ ohms}$$
$$R_c = \frac{R_2R_3}{R_1 + R_2 + R_3}$$
$$= \frac{20 \times 70}{100}$$
$$= 14 \text{ ohms}$$

c. **If the resistances are known for a wye circuit, they can be found for an equivalent delta circuit from the following equations:**

$$R_1 = \frac{R_aR_b + R_aR_c + R_bR_c}{R_a}$$
$$R_2 = \frac{R_aR_b + R_aR_c + R_bR_c}{R_b}$$
$$R_3 = \frac{R_aR_b + R_aR_c + R_bR_c}{R_c}$$

Example: **Find the equivalent resistance for a delta circuit if the resistances of a wye circuit are 10, 20, and 30 ohms.**

Let $R_a = 10$ ohms, $R_b = 20$ ohms, and $R_c = 30$ ohms.

$$R_1 = \frac{R_aR_b + R_aR_c + R_bR_c}{R_a}$$
$$= \frac{10 \times 20 + 10 \times 30 + 20 \times 30}{10}$$
$$= \frac{200 + 300 + 600}{10}$$
$$= \frac{1{,}100}{10}$$
$$= 110 \text{ ohms}$$
$$R_2 = \frac{R_aR_b + R_aR_c + R_bR_c}{R_b}$$
$$= \frac{1{,}100}{20}$$
$$= 55 \text{ ohms}$$
$$R_3 = \frac{R_aR_b + R_aR_c + R_bR_c}{R_c}$$
$$= \frac{1{,}100}{30}$$
$$= 36.666$$
$$= 36.67 \text{ ohms}$$

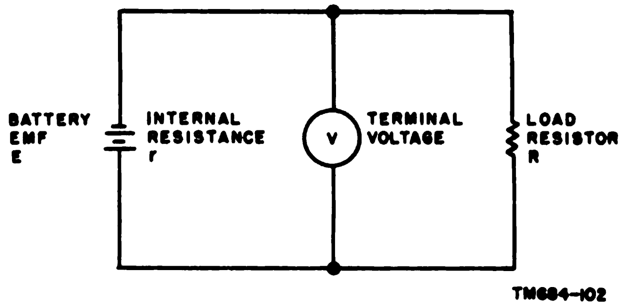


Figure 89. Maximum power transfer.

216. Maximum Power Transfer

a. Quadratic equations (par. 87–94) are used in problems involving the transfer of power from a source to a load. Such a problem can be illustrated by referring to figure 89. In the figure, the battery voltage is given as E, the internal resistance of the battery as r, the terminal voltage as V, and the load resistance as R. The total resistance of the circuit is $(R + r)$. From Ohm's law, $E = I(R + r)$, and the power delivered to the load is $P = VI$ where $V = E - Ir$.

b. The current through the circuit passes through the battery and drops the battery voltage to what is called the *terminal voltage*. Substituting for V in the power equation, $P = (E - Ir)I$, or $P = EI - I^2r$. A quadratic equation in I is obtained when the terms are rearranged. Thus, $-I^2r + EI - P = 0$. This equation can be solved for maximum current by using the method for finding the minimum value as a quadratic (par. 91). In the equation $-rI^2 + EI - P = f(I)$, $a = -r$, $b = E$, and $c = -P$. Substituting in the equation $I = \frac{-b}{2a}$, $I_{\text{max power}} = \frac{-E}{2(-r)} = \frac{E}{2r}$. This equation will give the current through the circuit when maximum power is delivered.

Example: If a 12-volt battery has an internal resistance of 3 ohms, find the current flowing in the circuit when maximum power is being delivered to the load.

$$\begin{aligned} I &= \frac{E}{2r} \\ &= \frac{12}{2 \times 3} \\ &= 2 \text{ amperes} \end{aligned}$$

c. If the value for current at maximum power transfer $\left(\frac{E}{2r}\right)$ is substituted in the original equation $I(R + r) = E$, a relationship between the load resistance and the internal resistance of the battery for maximum power transfer can be derived as follows:

$$\begin{aligned} I(R + r) &= E \\ \frac{E}{2r}(R + r) &= E \\ (R + r) &= 2r \qquad \text{(dividing by } E \text{ and multiplying by } 2r\text{)} \\ R &= 2r - r \\ R &= r \end{aligned}$$

Consequently, to obtain the maximum power transfer from the source to the load, the value of the load resistance must be equal to the internal resistance of the source.

217. Review Problems—Miscellaneous Electrical Problems

a. A generator is rated at 2,000 watts with a maximum output of 2,100 watts. What is the percent of overload capacity?

b. If the power input of a rotary converter is 48,000 watts and the power output is 37,800 watts, what is the efficiency?

c. The output of a generator is increased from 2,560 watts to 2,944 watts. How much is the increase when expressed in percent?

d. A 12,000-ohm resistance has a tolerance of plus or minus 5%. What is the maximum possible resistance?

e. If the input of an electric motor is 860 watts and the output is 746 watts, what is the efficiency of the motor?

f. A generator is rated at 2,000 watts and has a 10% overload capacity. What is the maximum output of the generator?

g. If the inductance L is 80 henrys and the energy P stored in the circuit is 100 joules, find the current I in amperes.

CHAPTER 18

GRAPHICAL REPRESENTATION AND SOLUTION OF ELECTRICAL PROBLEMS

218. Constructing and Reading Engineering Graphs

a. Constructing Graphs.

(1) Engineering graphs of operational or experimental data are constructed in the same manner as graphs of equations (pars. 100–102). First a chart is compiled of the available data, and then the data is plotted on an axis. The *independent variable* (the variable to which values are assigned) usually is plotted on the *x axis*, and the *dependent variable* on the *y axis*. The scales on the axes should be as large as practicable and, at the same time, keep the graph within the space available. Sometimes it may be convenient to choose a unit length for the ordinate different from that of the abscissa. Before selecting the units on the axes, examine the table for the maximum and minimum values of the variables and then choose the units on the axes to fit these values in the space available for the graph. Number the points at uniform intervals along the length of each axis, and label each scale.

(2) As an example, an experiment is conducted to determine the plate current (I_p) of a 6J5 electron tube at various values of plate voltage (E_p) when the tube has a grid bias of —6 volts. The plate voltages applied range from 120 to 235 volts. The information is tabulated as follows:

E_p	120	140	158	170	180	190	200	210	218	225	235
I_p	.5	1	2	3	4	5	6	7	8	9	10

(3) The plate current is then plotted against the plate voltage and the points joined by a smooth curve. The resulting graph (fig. 90) is a picture of the plate current-voltage characteristic of a 6J5 electron tube with a grid bias of —6 volts.

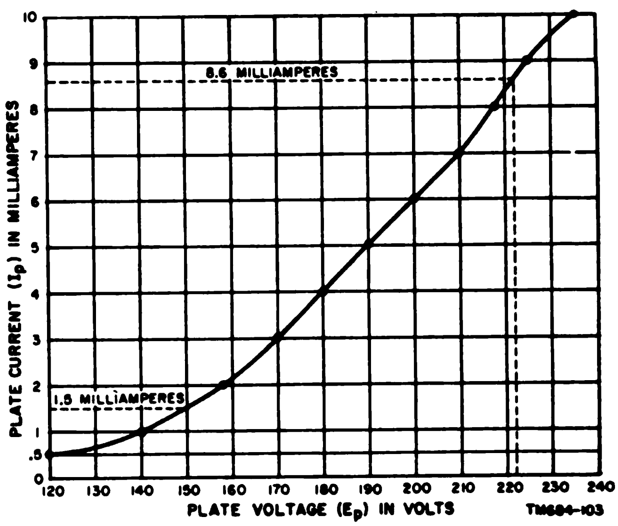


Figure 90. Graph showing plate current versus plate voltage characteristics of 6J5 electron tube with grid bias of —6 volts.

b. Reading Graphs. The process of finding properties of a function by inspection of the graph representing it is called *reading the graph*. From a study of the graph in figure 90, certain information is evident and additional information can be easily obtained.

(1) There is a gradual increase in the plate current of the 6J5 tube when the plate voltage is increased from 120 to 158 volts. From 158 to 210 volts, the current increase is fairly

steady. From 210 to 225 volts, however, the increase is sharper, but flattens out slightly from 225 to 235 volts.

(2) To determine the plate voltage that must be applied to result in a plate current of 8.6 milliamperes, draw a horizontal line from the 8.6 point on the *y axis* to the curve (fig. 90). At the point where this horizontal line intercepts the curve, drop a vertical line to the *x axis*. The required plate voltage is 222 volts. Similarly, to obtain a plate current of 1.5 milliamperes, a plate voltage of 150 volts must be applied.

219. Application of Graphs to Electrical Laws

a. *Example 1.*

(1) A variable resistance is connected across a generator that maintains a potential of 120 volts (fig. 91). The problem is to plot the current as the resistance is varied in 2-ohm steps from 0 to 20 ohms. Ohm's law, $I = \frac{E}{R}$, is used to obtain the coordinates. The voltage E is constant, the resistance R is the independent variable, and the current I is the dependent variable; thus, current will be plotted against resistance, and the independent variable, resistance, will be plotted along the *x axis*. The following chart is compiled:

R	0	2	4	6	8	10	12	14	16	18	20
I	∞	60	30	20	15	12	10	8.5	7.5	6.6	6

(2) After a study of the table, it will be found that it is more convenient to use a much smaller unit of measurement on the *x axis* than on the *y axis*. Also, the entire graph falls in the first quadrant as all values are positive. The resulting graph (fig. 92) is the current-resistance characteristic of the circuit. Note that the current decreases as the resistance increases. The current for any value of the variable resistance can be found by reading the graph.

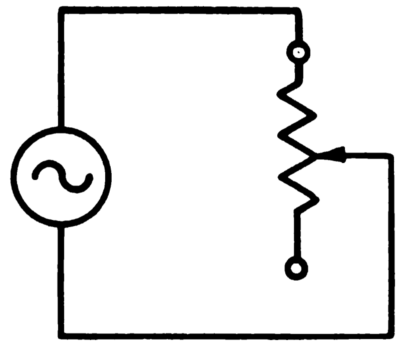


Figure 91. Series circuit showing variable resistance connected across generator.

b. *Example 2.*

(1) Figure 93 shows an ac series circuit with a coil having a fixed inductance connected across an ac generator that can be varied in frequency from 100 to 1,000 cps in steps of 100 cycles. It is assumed that the effect of the inductance L is so much greater than the resistance of the coil that the effect of the resistance can be neglected. The problem is to plot inductive reactance X_L in the formula $X_L = 2\pi fL$. The frequency f is varied to determine the effect upon the inductive reactance. L is constant at 0.04 henry, and 2π equals 6.28; thus, inductive reactance will be plotted against frequency, with the frequency plotted along the *x axis*. The following chart is compiled:

f	100	200	300	400	500	600	700	800	900	1,000
X_L	25.1	50.2	75.4	100.5	125.5	150.7	175.8	201.0	226.1	251.2

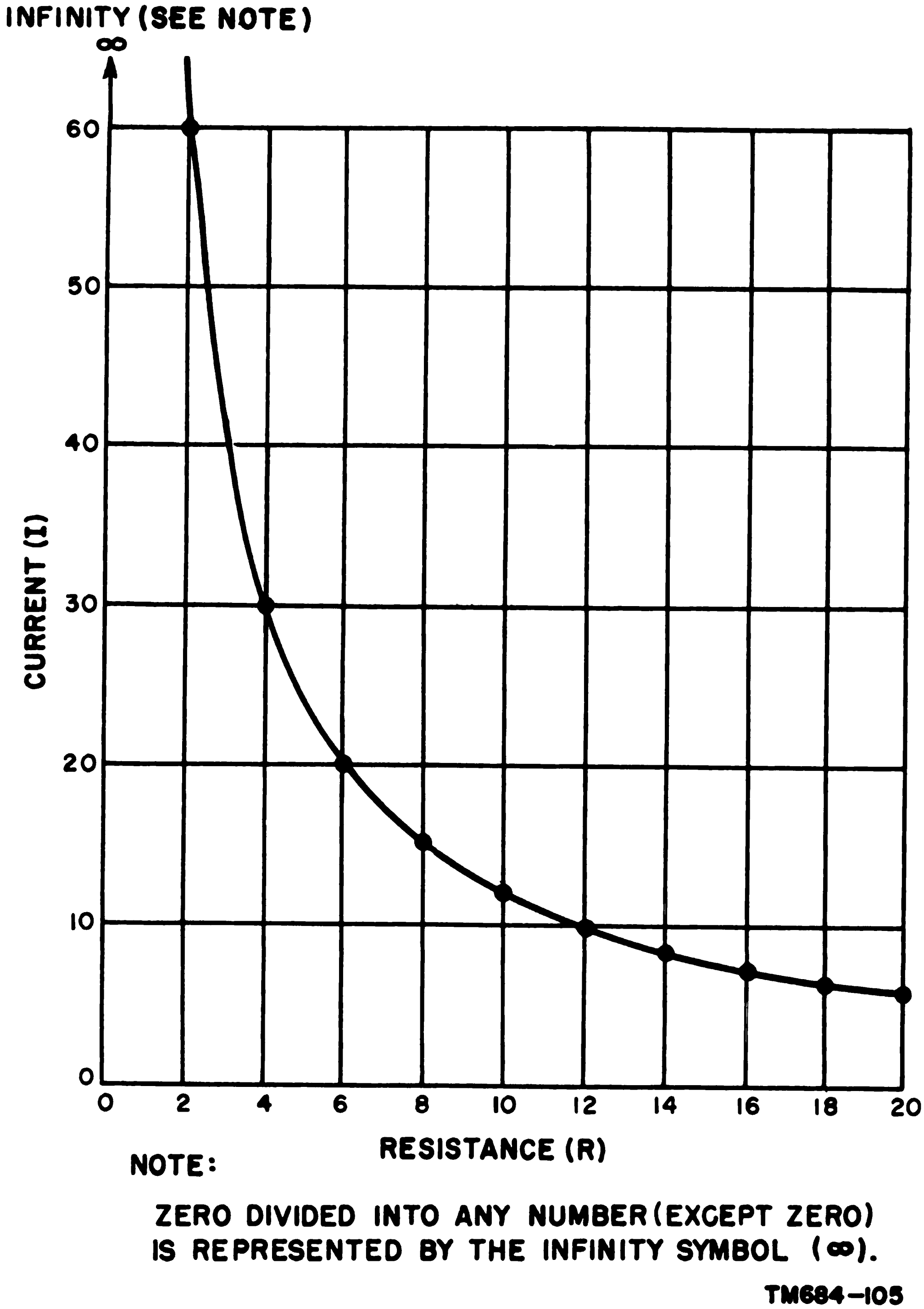


Figure 92. Graph showing current versus resistance curve for series circuit with 120-volt potential.

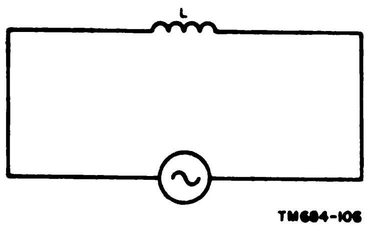


Figure 93. Series circuit showing inductance connected across ac generator.

(2) Since all values are positive, the entire graph will lie in the first quadrant. The resulting graph (fig. 94) pictures the increase in the reactance of the inductor as operating frequencies are increased from 100 to 1,000 cycles per second.

220. Review Problems—Graphical Representation and Solution of Electrical Problems

a. The antenna resistance *R* in ohms varies as the height *H* in wavelengths of a horizontal half-wave antenna according to the values given in the chart below. Plot the curve of the antenna resistance against the antenna height. At what height is the resistance at a maximum?

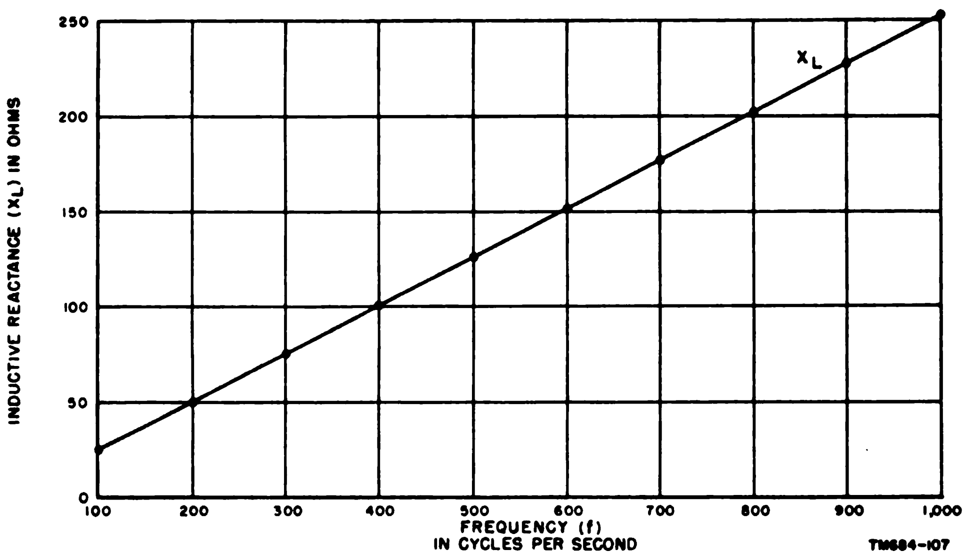


Figure 94. Graph showing reactance of 0.4-henry inductor at frequencies from 100 to 1,000 cps.

H	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
R	0	30	87	95	68	60	77	85	68

b. As the output current I_o in milliamperes is varied by a full-wave rectifier voltage quadrupler, the output voltage E_o in volts changes in accordance with the following data. Plot the curve and determine the current at a voltage of 380 volts.

I_o	45.5	42.0	39.5	36.0	32.5	28.5	24.0	19.5	14.0	8.0	4.0
E_o	292	305	317	330	350	370	390	415	448	488	515

c. When two coils are arranged so that a change in current in one coil causes a voltage to be induced in the other, the coils are said to possess mutual inductance. Given the mutual inductance *M* in henrys for two coils *S* centimeters apart, plot the curve of the mutual inductance against the separation between the coils. What is the mutual inductance when the coils are separated by a distance of 7 centimeters?

S	0	2	4	6	8	10	12	14
M	0.051	0.049	0.041	0.033	0.025	0.017	0.011	0.007

d. The vertical sag S in a powerline depends on the temperature T. With the sag being measured in feet and the temperature in °F. the following data is available for a 400-foot span. Plot the vertical sag against the temperature. If the sag is not to exceed 8.1 feet, what is the maximum permissible temperature?

T	—40	—20	0	20	40	60	80	100
S	6.8	7.0	7.2	7.4	7.6	7.8	8.0	8.2

e. The values of current I in milliamperes obtained by applying E volts to a selenium rectifier plate is shown in the following chart. Plot the current against the voltage and determine the current when the voltage is 0.8 volt.

E	1.5	1.3	1.1	0.9	0.7	0	—2	—4	—6	—8	—10
I	100	80	60	40	20	0	—.05	—1	—2	—3	—4

f. Using the formula $XC = \frac{1,000,000}{2\pi fC}$ to determine the values of variables, plot a graph showing reactance XC of a circuit having a capacitance of 2 microfarads at frequencies f variable from 1,000 to 10,000 cps in 1,000-cycle steps. ($2\pi = 6.28$.)

g. A circuit consists of a resistance of 5 ohms connected across a source of variable potential. Using Ohm's law, plot the current through the resistance against the voltage across the resistance as the voltage is varied from 0 to 120 volts in 10-volt steps.

h. A variable resistance is connected across a generator that maintains a potential of 220 volts. Plot the current through the resistance as the resistance is varied in 5-ohm steps from 5 to 60 ohms.

CHAPTER 19

BINARY NUMBERS

221. Scope and Background

a. This chapter serves as an introduction to the theory and arithmetic of binary numbers. It explains the difference between binary numbers and the more conventional decimal numbering system.

b. Binary numbers are of primary interest to the electronic technician because of their use in digital computers and similar devices. These computers fundamentally depend on either a conducting or nonconducting state of vacuum tubes or transistors, or they may depend on the storage states of magnetic cores. Hence they are bistable; that is, they are in one of two stable conditions.

c. The decimal numbering system uses 10 digits, 0 through 9. A digital computer using the decimal system would be large and complex; hence the binary system was adapted for digital computer use—the two digits, 0 and 1, of the binary system correspond to the bistable states discussed in *b* above. These two digits are called *bits*, a contraction of *B*inary dig*ITS*.

222. Comparison Between Decimal and Binary Systems

a. Decimal System. In the decimal system, the value of a number depends on the position of its digits. For example, in the decimal number 63, the digit 3 represents 3; however, in 63,444, the digit 3 represents 3,000; thus, changing the place of a digit in a number changes the value of the digit.

(1) As another example, decimal number 825 means $8 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$. Ten (10) is considered the radix or base of the decimal system. Positional values for the equivalent powers of 10 are given in the following chart:

Position	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Units
Value	1,000,000	100,000	10,000	1,000	100	10	1
Power	10^6	10^5	10^4	10^3	10^2	10^1	10^0

(2) Again, using 63,444 as an example, the number can be analyzed as follows:

Position digit	6	3	4	4	4
Power	10^4	10^3	10^2	10^1	10^0

(3) Now multiply the position digit by the power (value) and add:

$$
\begin{aligned}
6 \times 10^4 &= 60{,}000 \\
3 \times 10^3 &= 3{,}000 \\
4 \times 10^2 &= 400 \\
4 \times 10^1 &= 40 \\
4 \times 10^0 \text{ (or 1)} &= +\ 4 \\
\hline
&\quad 63{,}444
\end{aligned}
$$

b. Binary System. In place of 10, the binary system uses 2 as the base or radix. All powers are powers of 2. An expansion (multiplication) of some of the powers of 2 follows:

$$
\begin{aligned}
2^0 &= 1 \\
2^1 &= 2 \\
2^2 &= 4 \\
2^3 &= 8 \\
2^4 &= 16 \\
2^5 &= 32 \\
2^6 &= 64 \\
2^7 &= 128 \\
2^8 &= 256\text{, etc.}
\end{aligned}
$$

(1) The binary system operates like the decimal system. Use the number 111111 as an example:

(*a*) In the decimal system:
$111111 = 10^5 + 10^4 + 10^3 + 10^2 + 10^1 + 10^0 = 111{,}111.$

(*b*) In the binary system:
$111111 = 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 63.$

(2) A portion of the positional values and equivalent powers used in the binary system is now given.

Position	Sixty-four	Thirty-two	Sixteen	Eight	Four	Two	Units
Value	64	32	16	8	4	2	1
Power	2^6	2^5	2^4	2^3	2^2	2^1	2^0

223. Tabular Conversion of Decimal Numbers to Binary Numbers

a. The following chart expresses the decimal numbers 0 through 10 in the binary system:

Decimal numbers	Binary numbers					
	2^5	2^4	2^3	2^2	2^1	2^0
	32	16		4	2	1
0	0	0	0	0	0	0
1	0	0	0	0	0	1
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	0	0	1	0	0
5	0	0	0	1	0	1
6	0	0	0	1	1	0
7	0	0	0	1	1	1
8	0	0	1	0	0	0
9	0	0	1	0	0	1
10	0	0	1	0	1	0

(1) If a power of 2 appears in the decimal number in the left column, place a 1 in the column in which the power of two appears. If a power of 2 is not used, place a 0 in that column.

(2) The decimal number 0 is equivalent to the binary number 0. Thus, a 0 is required in the extreme right-hand position of the binary system.

(3) The decimal number 2 equals 2^1—place a 1 under 2^1 and a 0 under all other powers of 2.

(4) The decimal number 3 equals $2^1 + 2^0$. Place a 1 in each of these columns and a 0 under all other powers of 2.

(5) The decimal number 4 equals 2^2. Place a 1 under 2^2 and a 0 under all other powers of 2.

(6) The decimal number 5 equals $2^2 + 2^0$. Place a 1 under each of these powers of 2 and a 0 under all of the remaining powers.

(7) Use the procedures outlined above to check the remaining values in the chart.

b. Additional tabular conversions follow:

Decimal numbers	Binary numbers					
	2^5	2^4	2^3	2^2	2^1	2^0
	32	16	8	4	2	1
20	0	1	0	1	0	0
30	0	1	1	1	1	0
40	1	0	1	0	0	0
45	1	0	1	1	0	1
50	1	1	0	0	1	0
57	1	1	1	0	0	1

224. Nontabular Conversion of Decimal Numbers to Binary Numbers

The tabular conversion of decimal numbers to binary numbers is tedious and somewhat awkward. An easier method is to divide the decimal number by 2, and the answer again by two, continuing until you have a remainder of 1. In the example below, 37 will be converted to its binary equivalent. Notice that throughout the operation all numbers will be either exactly divisible by 2 or will be divisible with a remainder of 1. If 2 divides evenly, place a

0 to the right of that quotient; if 2 does not divide evenly, place a 1 to the right of that quotient; repeat until further division by 2 is impossible.

Example:

```
2/37
2/18  1
2/ 9  0
2/ 4  1
2/ 2  0
2/ 1  0
   0  1
```

The binary number, 100101, is obtained by reading from *bottom to top*. This result may be checked against the tabular system of conversion (par. 257).

225. Tabular Conversion of Binary Numbers to Decimal Numbers

Using the binary number 1011010, the following procedure illustrates one method of converting from binary numbers to decimal numbers:

Bit position	1	0	1	1	0	1	0
Power value	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Multiply bit position times power value	64	0	16	8	0	2	0
Add horizontally	64 +	0 +	16 +	8 +	0 +	2 +	0
Total	90,						
The decimal equivalent, therefore, is 90.							

226. Nontabular Conversion of Binary Numbers to Decimal Numbers

The following procedure illustrates an alternative method of coverting from binary numbers to decimal numbers—the same binary number, 1011010 is used:

a. Start with the bit at the extreme left.

(1) If the next bit to the right is a 0, double the leftmost bit.

(2) However, if this next bit is a 1, double the leftmost bit and add 1.

b. The complete conversion of 1011010 follows:

(1) At the first left bit, double 1 to get 2 since the second bit from left is a 0.

(2) At the second lift bit, double 2 and add 1 to get 5 since the third bit from left is a 1.

(3) At the third left bit, double 5 and add 1 to get 11 since the fourth bit from the left is a 1.

(4) At the fourth bit, double 11 to get 22 since the fifth bit from the left is a 0.

(5) At the fifth bit, double 22 and add 1 to get 45 since the sixth bit is a 1.

(6) At the sixth bit, double 45 to get 90 since the seventh bit is a 0. This is the answer and the end of the operation.

(7) Since the seventh bit is the last bit, no further operations are required. *Remember that no mathematical operation is required for the extreme right-hand bit when converting by the nontabular method.*

227. Addition of Binary Numbers

a. Addition in the binary system is similar to addition in the decimal system. The rules for binary addition follow:

(1) 0 + 0 = 0

(2) 0 + 1 = 1

(3) 1 + 1 = 10, 0 with 1 to carry into the next place. This rule may be expanded further to include: 1 + 1 = 11, or 1 with 1 to carry to the next place. 1 + 1 + 1 + 1 = 100, or 0 with 10 to carry in the next place.

b. The following example illustrates binary addition.

Binary	*Decimal*
1101	13
+ 1111	15
11100	28

(1) Begin at the extreme right bit: 1 + 1 = 10. Write 0, carry 1.

(2) 0 + 1 + 1 (carried over) = 10. Write 0, carry 1.

(3) 1 + 1 + 1 (carried over) = 11. Write 1, carry 1.

(4) 1 + 1 + 1 (carried over) = 11. Write 11.

(5) The answer is 11100. Check answer by converting to decimal numbers and then adding (as shown).

c. Binary addition is further illustrated by the next example:

Binary	*Decimal*
101 1101 1101	1501
+ 111 0010 1101	+1837
1101 0000 1010	3338

228. Subtraction of Binary Numbers

a. Subtraction in the binary system is similar to subtraction in the decimal system. The rules for binary subtraction follow:

(1) 0 — 0 = 0

(2) 1 — 1 = 0

(3) 1 — 0 = 1

(4) 0 — 1 = 1, and then proceed to change all numbers in the top row until you change a 1 to a 0.

b. The following example illustrates binary subtraction:

Binary		*Decimal*
1011	(minuend)	11
—0101	(subtrahend)	5
0110	(remainder)	6

(1) Begin at the extreme right bit: 1 — 1 = 0. Bring down 0, none to carry.

(2) 1 — 0 = 1. Bring down 1, none to carry.

(3) 0 — 1 = 1. The bit 1 to the extreme left in the minuend is changed to 0.

(4) 0 — 0 = 0. End of operation.

(5) The complete calculation may be checked by adding the subtrahend and the remainder.

Binary	*Decimal*
0101	5
+0110	+ 6
1011	11

c. Binary subtraction is further illustrated in the next example:

Binary		*Decimal*
1101 0000 1010	(minuend)	3338
— 111 0010 1101	(subtrahend)	—1837
101 1101 1101	(remainder)	1501

Proof:

Binary	*Decimal*
101 1101 1101	1501
+ 111 0010 1101	+1837
1101 0000 1010	3338

229. Complementary Addition of Binary Numbers

The direct subtraction of binary numbers is not used in some data equipments. Instead, the subtraction processes are carried out by complement addition. To subtract two binary numbers using this system, proceed as follows:

a. Use the following problem as an example:

1101101	(minuend)
— 10010	(subtrahend)

b. First determine the complement of the subtrahend.

(1) Add zeros to the left until the subtrahend has the same number of bits as the minuend; for example, 0010010.

(2) Note the first 1 counting from the right and bring down this 1 as well as any of the zeros to its right; then reverse all other bits proceeding toward the left. The subtrahend of (1) above becomes *1101110*. This number is the complement.

(3) Now add the minuend to the complement:

```
    110 1101
  + 110 1110
   1101 1011
```

(4) Delete the 1 to the extreme left and the remainder becomes 1011011.

(5) Check the answer by binary subtraction:

```
  110 1101   (minuend)
—   1 0010   (subtrahend)
  101 1011   (remainder)
```

c. The next problem is solved by using the principles explained in *a* and *b* above.

(1) 11101101 (minuend)
  111100 (subtrahend)

(2) 00111100 becomes the complement 11000100.

(3)
```
   1110 1101   (minuend)
+  1100 0100   (complement)
  ⁄11011 0001   (remainder)
```

(4) Proof by subtraction

```
  11101101   (minuend)
—   111100   (subtrahend)
  10110001   (remainder)
```

230. Multiplication of Binary Numbers

a. Multiplication is the simplest of all the binary processes. The rules are:

(1) $0 \times 0 = 0$

(2) $0 \times 1 = 0$

(3) $1 \times 1 = 1$

b. Remember that binary *addition* is important to binary multiplication. Two examples of multiplication are given below.

Example 1:

```
Binary       Decimal
  1011         11
×   10        × 2
  0000
 1011
 10110         22
```

Example 2:

```
Binary          Decimal
   111011         59
×     101        × 5
   111011
  000000
 111011
100100111        295
```

231. Division of Binary Numbers

a. Division of binary numbers is similar to division in the decimal system. The simple rules are:

(1) $0 \div 0 = 0$

(2) $0 \div 1 = 0$

(3) $1 \div 1 = 1$

b. Remember that binary subtraction is important to binary division. Two examples of binary division are given below.

Example 1:

```
Binary          Decimal
     111            7
 10)1110         2)14
    10             14
     11
     10
      10
      10
```

Proof:

```
Binary          Decimal
  111              7
×  10            × 2
 1110             14
```

Example 2:

```
Binary                  Decimal
       1001 100/110        9 4/6
110)111010               6)58
     110                   54
       1010                4/6
        110
        100
```

Note the remainder of 100/110.

Proof:

```
  1001    (partial quotient)
× 110     (divisor)
 10010
1001
110110    (partial dividend)
  +100    (add remainder)
111010    (total dividend)
```

232. Fractions in the Binary System

a. The system of expressing fractions with binary numbers is similar to the decimal numbering methods. For example, the common fraction $\frac{3}{5}$ may be expressed in binary numbers as $\frac{11}{101}$ Also, binary fractions may be expressed as decimal fractions when the powers of 2 are used with negative exponents. The binary fraction 0.011 is equivalent to the decimal fraction 0.375 and may be written as:

$$0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} = 0 + \frac{1}{4} + \frac{1}{8} = \frac{3}{8} \text{ or } .375.$$

b. The following table lists some of the fractional values and their equivalents in both systems:

Decimal equivalents	*Power of 2*	*Binary equivalent*
$\frac{1}{2}$ or .5	2^{-1}	.1
$\frac{1}{4}$ or .25	2^{-2}	.01
$\frac{1}{8}$ or .125	2^{-3}	.001
$\frac{1}{16}$ or .065	2^{-4}	.0001
$\frac{1}{32}$ or .03125	2^{-5}	.00001
$\frac{1}{64}$ or .015625	2^{-6}	.000001
$\frac{1}{128}$ or .0078125	2^{-7}	.0000001
$\frac{1}{256}$ or .00390625	2^{-8}	.00000001
$\frac{1}{512}$ or .001953125	2^{-9}	.000000001

c. Using values from the table, the decimal fraction 0.375 is equal to .25 + .125 and hence has the binary equivalency of .01 + .001 = .011.

233. Conversion of Decimal Fractions to Binary Numbers

Usually, the decimal fractions are converted to binary fractions by performing a series of multiplications by 2. This method is directly opposite to the method explained in paragraph 224. As a rule, decimal fractions cannot be converted to exact binary equivalents. The extent of error must be tolerable for a given application and the number of bits used must be reasonable.

a. To convert 0.375 to a binary number, proceed as follows:

(1) Multiply the decimal 0.375 by 2 to obtain a new integer (whole number) and a new decimal, 0.75. Since in 0.75 the integer to the left of the decimal point is 0, place a 0 in the binary equivalent as .0.

Decimal × 2	New integer and decimal	Partial binary equivalent
0.375 × 2	0.75	.0

(2) Multiply the decimal 0.75 by 2 to obtain a new integer and decimal. Since the integer to the left of the decimal point is a 1, place a 1 in the binary equivalent as .01.

Decimal × 2	New integer and decimal	Partial binary equivalent
0.75 × 2	1.50	.01

(3) Drop the integer 1 and multiply the decimal 0.50 by 2 to obtain a new integer and decimal. Since the new integer to the left of the decimal is a 1, place another 1 in the binary equivalent as .011.

Decimal × 2	New integer and decimal	Partial binary equivalent
0.50 × 2	1.00	.011

(4) Note that the operation ends when the decimal part has been expanded to 0.00. The decimal fraction, 0.375, is equivalent to the binary fraction, .011. In this instance, the binary and decimal fractions have exactly the same value.

b. The next example illustrates the conversion of 0.3465 to its binary equivalent. Note that the partial binary equivalents are added at the end of the operation to obtain the complete equivalent:

Multiplication x 2	*Binary equivalent*	*Decimal value of binary equivalent*
A	B	C
0.3465 × 2 = .6930	.0	.0
0.693 × 2 = 1.386	.01	.25
0.386 × 2 = 0.772	.000	.00
0.772 × 2 = 1.544	.0001	.0625
0.554 × 2 = 1.108	.00001	.03125
Add up all entries under B and C.	.01011	.34375

c. In *b* above, the binary and decimal fractions differ in value and the amount of error may be determined by subtraction:

0.3465 — 0.34375 = 0.00275 (fraction of error)

234. Mixed Binary Numbers

a. A mixed binary number is a combination of whole numbers and binary fractions. Examples of this are:

Binary number	*Decimal equivalent*
1011.1	11.5
1110.011	14.375
10.000001	1.015625

b. The fundamental operations (addition, subtraction, multiplication, and division) for mixed binary numbers or binary fractional numbers alone are in accordance with the principles already explained in this chapter.

APPENDIX I

BASIC SLIDE RULE OPERATIONS

1. General

This appendix describes the basic slide rule and covers the operations of multiplication, division, squaring, and square root.

2. Description of Slide Rule

a. Slide rules are made in several different sizes and styles, and in an assortment of scales. However, they all contain the same basic scales and use them in the same manner.

b. The most common type of slide rule is about 10 inches long and generally has scales on both sides. The most frequently used scales, and the ones covered here, are the A, B, C, and D. Figure 95 is a simplified drawing of a slide rule of this type, showing these scales and the other essential parts of the rule. Note that these scales have indexes (the number 1) on both ends. Also note that the A and B scales have an additional index in the center that divides these scales into two equal parts. The left-hand part of the scales is called A-left or B-left, and the right-hand part, A-right or B-right.

3. Basic Principles of Operation

The slide rule is based on the principle of the logarithm; that is, the segments on the rule represent exponents, or logarithms, but are indicated by the antilogs, or numbers corresponding to those logarithms. Consequently, when the slide rule is used so that two line segments are added, the logarithms of the numbers shown are actually being added, and the sum of the two line segments is represented by the antilog of the sum of the logarithms. Since the sum of the logarithms of two numbers is equal to the logarithm of the product of the two numbers (par. 121), adding two line segments on a slide rule will give the product of the two numbers represented by the line segments. This is the technique used in multiplication with a slide rule (par. 6 of this app). In the division process, the reverse procedure is used; that is the two line segments are used so that one is subtracted from the other.

4. Accuracy

The accuracy of a slide rule depends on the length of the rule and on the portion of the rule being used. With the 10-inch rule shown in figure 95, numbers can be approximated to four significant figures on the left-hand end of the C or D scales, but only to three significant figures on the right-hand end of these scales. Despite this fact, the results obtained with the slide rule are sufficiently accurate for many practical purposes; in any case, the results serve as a rapid and efficient check of more complex computations.

5. Reading the Scales

a. Since the scales on a slide rule do not have uniform increments along their lengths,

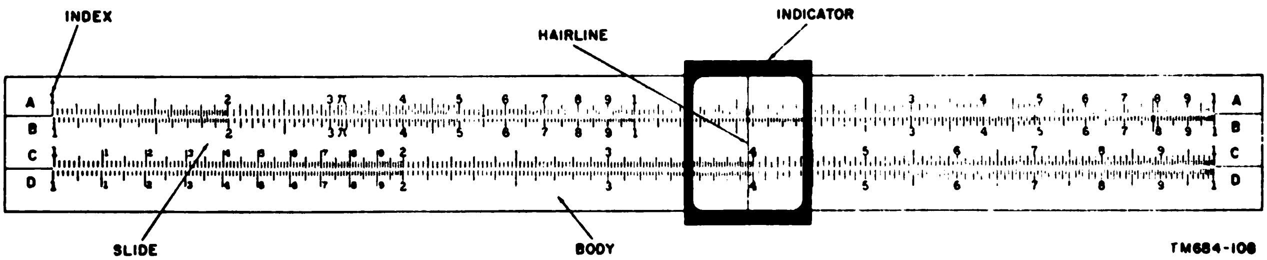


Figure 95. Typical slide rule, simplified drawing.

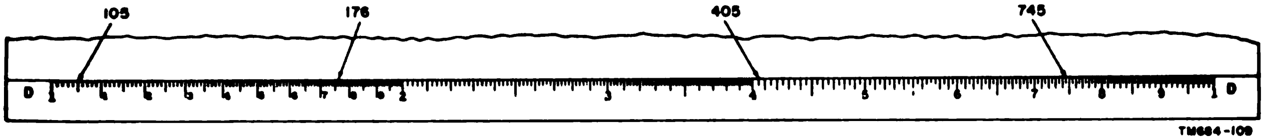


Figure 96. Locating numbers on the D scale.

be careful when approximating numbers at different points on the scales. For example, the space between the larger numbers 1 and 2 on the D scale (fig. 96) is divided into 10 subgroups (identified by the small numbers 1 through 10) of 10 increments each; thus there are 100 increments between 1 and 2 on the D scale, and each increment is equal to one one-hundredth of the difference. Between 4 and 5 on the D scale, however, there are only 20 increments, and each increment therefore, is equal to five one-hundredths of the difference. Consequently, the number 105 would be located 5 increments above 1, whereas 405 would be 1 increment above 4 on the scale. Figure 96 shows the location of these and other numbers on the D scale.

b. To locate a number on a scale, first determine its general location between two of the numbers on that scale; then determine the value of each increment between the numbers. Finally, determine its exact location based on the value of the increments.

c. In reading the scale, as in logarithms, the decimal point is neglected until after the absolute value of the result is obtained; therefore, in figure 96, the number 1245 could actually represent 1.245, 12.45, 124.5, .001245, etc. The use of scientific notation (par. 106) will greatly simplify the handling of very large or very small numbers.

6. Multiplication

a. Normally, the process of multiplication is performed by using the C and D scales. The A and B scales may also be used, but they are not as accurate because the increments are smaller. To multiply two numbers, proceed as follows:

(1) Locate one number on the D scale. Slide the indicator until the hairline is over the number to mark its location.

(2) Place one of the indexes of the C scale above the number on the D scale. Use the hairline of the indicator to aline the index and the number.

(3) Locate the second number on the C scale. If the number is located on the portion of the C scale beyond the end of the D scale, reposition the slide so that the other index on the C scale is above the number on the D scale.

(4) Slide the indicator so that the hairline is over the number on the C scale. The product of the two numbers is read under the hairline on the D scale.

b. The two examples below illustrate the method of multiplication described above. They also point out the use of the two indexes on the C scale.

Example 1: Multiply 2 by 3.

Step 1. Locate the number 2 on the D scale and slide the indicator until the hairline is over it.

Step 2. Place the left-hand index of the C scale above the number 2 on the D scale. Use the hairline on the indicator for alinement.

Step 3. Locate the number 3 on the C scale and slide the indicator so that the hairline is over it. The

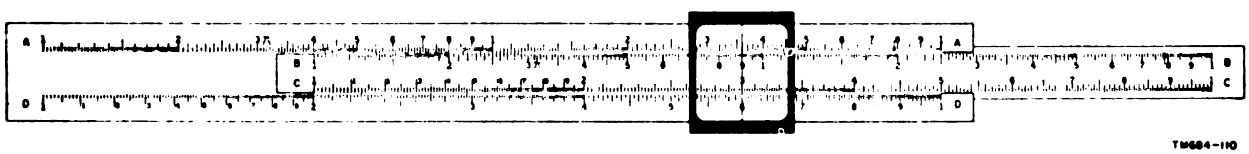


Figure 97. Slide rule arranged for multiplying 2 by 3.

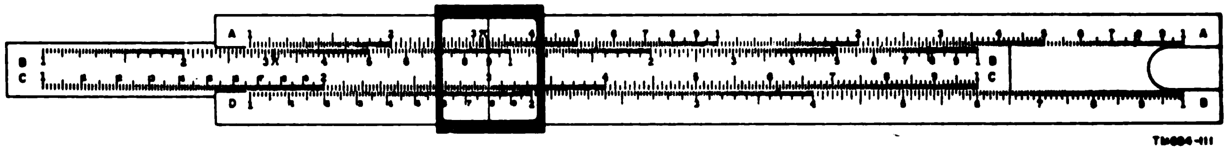


Figure 98. Slide rule arranged for multiplying 6 by 3.

product of 2 times 3 or 6 is read under the hairline on the D scale. Figure 97 shows a slide rule arranged for this product.

Example 2: Multiply 6 by 3.

Step 1. Locate the number 6 on the D scale and slide the indicator so that the hairline is over it.

Step 2. Place the right-hand index of the C scale above the number 6 on the D scale. Use the hairline on the indicator for alinement. (The right-hand index is used because the number 3 on the C scale would be beyond the end of the D scale if the left-hand index were used.)

Step 3. Locate the number 3 on the C scale and slide the indicator so that the hairline is over it. The product of 6 times 3 or 18 is read under the hairline on the D scale. Figure 98 shows a slide rule arranged for this product.

7. Division

a. The process of division, like multiplication, generally is performed by using the C and D scales. To divide one number by another number, proceed as follows:

(1) Locate the dividend (number to be divided) on the D scale. Slide the indicator until the hairline is over the number to mark its location.

(2) Locate the divisor on the C scale. Move the slide until this number is above the dividend on the D scale. Use the hairline on the indicator for alinement.

(3) Slide the indicator until the hairline is over the index on the C scale that is above a portion of the D scale. The quotient of the two numbers is read under the hairline on the D scale.

b. The following example illustrates the use of the method of division described above.

Example: Divide 8 by 2.

Step 1. Locate the dividend (8) on the D scale and slide the indicator until the hairline is over it.

Step 2. Locate the divisor (2) on the C scale and move the slide until this number is above 8 on the D scale. Use the hairline on the indicator for alinement.

Step 3 Slide the indicator until the hairline is over the left-hand index. The quotient of 8 divided by 2 is located under the hairline on the D scale. Figure 99 shows a slide rule arranged for this quotient.

8. Squaring a Number

a. The process of squaring a number is performed by using the A and D scales. To square a number, proceed as follows:

(1) Locate the number on the D scale. Slide the indicator until the hairline is over the number.

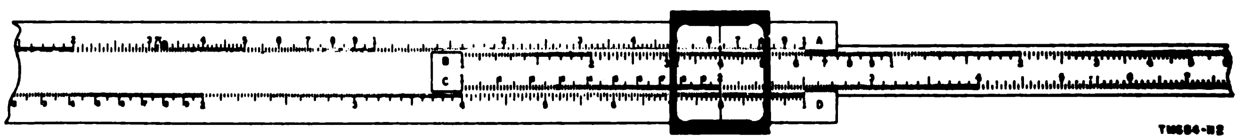


Figure 99. Slide rule arranged for dividing 8 by 2.

(2) Read the square under the hairline on the A scale. Remember that the increments on the A scale are smaller than the increments on the D scale; be sure to evaluate the increment carefully.

b. The following example illustrates the procedure given above.

Example: Find the square of 12.5.

Step 1. Locate 12.5 on the D scale. Slide the indicator until the hairline is over it (fig. 100).

Step 2. Read the square of 12.5 under the hairline on the A scale. The three significant figures that can be obtained from the A scale are 156. To locate the decimal point, estimate the value of the square of 12.5. Since the square of 12 is 144, the square of 12.5 is 156.

9. Square Root of a Number

a. The process of finding the square root of a number is simply the reverse of the process of squaring a number (par. 8). To find the square root of a number, proceed as follows:

(1) Locate the number of the A scale. Slide the indicator until the hairline is over the number.

(2) Read the square root under the hairline on the D scale.

b. The following example illustrates the procedure given above.

Example: Find the square root of 9.5.

Step 1. Locate 9.5 on the A scale. Note that there are two such numbers on the A scale, one on A-left, and one on A-right. Since the square root of 9 is 3, the number on A-right would not yield the correct result; therefore, slide the indicator until the hairline is over the 9.5 on A-left.

Step 2. Read the square root of 9.5, that is, 3.08, under the hairline on the D scale. Figure 101 shows a slide rule arranged for this square root.

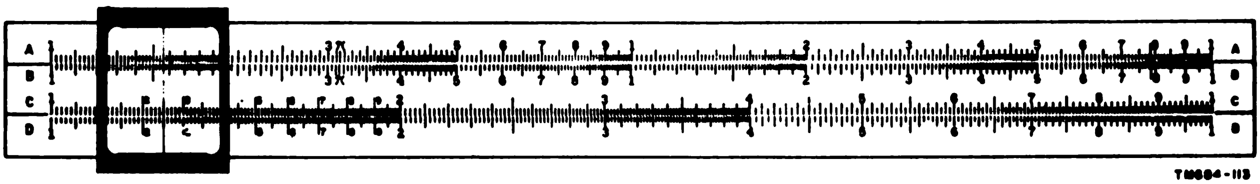


Figure 100. Slide rule arranged for finding the square of 12.5.

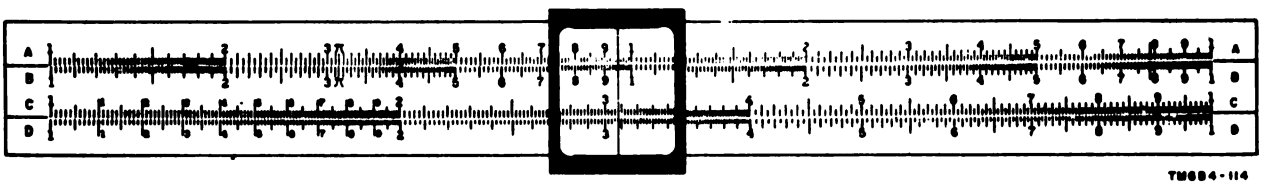


Figure 101. Slide rule arranged for finding the square root of 9.5.

APPENDIX II
SYSTEMS OF MEASUREMENT

1. General

Two systems of measurement are in use in the United States today: the English system, based on the foot and the pound, and the metric system, based on the centimeter (or meter) and the gram (or kilogram). Both systems are used in electronics. For example, the wavelength of an antenna is calculated in the metric system; the physical length of each conductor is stated in feet and inches.

2. Metric Prefixes

In the field of communications, there are often wide ranges in electrical quantities. For example, the input of a radio receiver may be in millionths of a volt, and the output circuit of a transmitter may be in thousands of volts. Thus, metric prefixes are used in electronics in combination with basic units of measurement—volts, ohms, watts, amperes, farads, henrys, and cycles—to facilitate operations. The following chart gives the meaning of these prefixes with respect to various units of measurement.

Metric prefix	Meaning		Associated with
Mega	Million	(1,000,000)	Volt, ohms, cycles, amperes
Kilo	Thousand	(1,000)	Volts, watts, cycles, meters, amperes
Hecto	Hundred	(100)	Meters
Deka	Ten	(10)	Meters
Deci	One-tenth	(0.1)	Meters
Centi	One-hundredth	(0.01)	Meters
Milli	One-thousandth	(0.001)	Volts, amperes, meters, henrys, watts, ohms
Micro	One-millionth	(0.0000001)	Volts, amperes, farads, henrys, mhos, ohms
Micromicro	One-millionth of one-millionth	(0.000,000,000,001)	Volts, amperes, farads, coulombs

3. Conversion Factors

The table below lists the common units of measurement with one set of prefixes and the factor by which these units must be multiplied to convert them to units with another set of prefixes. The examples below illustrate the method in which the table is used.

Example 1: Convert 7.54 megacycles to cycles.

From the table, to convert from megacycles to cycles, multiply by 10^6.

$7.54 \times 10^6 = 7.54 \times 1{,}000{,}000 = 7{,}540{,}000$ cycles

Example 2: Convert 5,500 watts to kilowatts.

From the table, to convert watts to kilowatts, multiply by .001.

$5{,}500 \times 0.001 = 5.5$ watts

To convert from	To	Multiply by
Abamperes	Amperes	10.0000
Abamperes	Statampere	2.998×10^{10}
Abcoulombs	Ampere-hours	2.778×10^{-3}
Abcoulombs	Coulombs	10.0000
Abcoulombs	Faradays	1.036×10^{-4}
Abcoulombs	Statcoulombs	2.998×10^{10}
Abfarads	Farads	10^{9}
Abfarads	Microfarads	10^{15}
Abfarads	Statfarads	8.988×10^{20}
Abhenrys	Henrys	10^{-9}
Abhenrys	Microhenrys	.001
Abhenrys	Millihenrys	10^{-6}
Abhenrys	Stathenrys	$1,113 \times 10^{-21}$
Abohms	Megohms	10^{-15}
Abohms	Microhms	0.001
Abohms	Ohms	10^{-9}
Abohms	Statohms	1.113×10^{-21}
Abvolts	Microvolts	.01
Abvolts	Millivolts	10^{-6}
Abvolts	Statvolts	3.336×10^{-1}
Abvolts	Volts	10^{-8}
Acres	Ares (square dekameters)	40.46873
Acres	Hectares (square hectometers)	.4046873
Acres	Square feet	4.356×10^{4}
Acres	Square inches	6,272,640
Acres	Square kilometers	4.047×10^{-3}
Acres	Square meters	4047
Acres	Square miles	1.563×10^{-6}
Acres	Square rods	160
Acres	Square yards	4840
Amperes	Abamperes	.1
Amperes	Milliamperes	1000
Amperes	Statamperes	2.998×10^{9}
Ampere-hours	Abcoulombs	360
Ampere-hours	Coulombs	3600
Ampere-hours	Faradays	3.731×10^{-2}
Ampere-hours	Statcoulombs	1.080×10^{13}
Ares	Acres (US)	.02471044
Ares	Hectares	.01
Ares	Square feet	1076.4
Ares	Square meters	100
Ares	Square miles	3.861×10^{-6}
Ares	Square yards	119.60
Bushels (dry)	Cubic centimeters	3524×10^{4}
Bushels (dry)	Cubic feet	1.2444
Bushels (dry)	Cubic inches	2150.4
Bushels (dry)	Cubic meters	3.524×10^{-2}
Bushels (dry)	Liters	35.24
Centimeters	Feet	3.281×10^{-2}
Centimeters	Inches	.3937
Centimeters	Kilometers	10^{-5}
Centimeters	Meters	.01
Centimeters	Mils	393.7
Centimeters	Miles	6.214×10^{-6}
Centimeters	Millimeters	10
Centimeters	Yards	1.094×10^{-2}
Centimeters/second	Feet/minute	1.969
Centimeters/second	Feet/second	3.282×10^{-2}
Centimeters/second	Kilometers/hour	.036

To convert from	To	Multiply by
Centimeters/second	Kilometers/minute	.0006
Centimeters/second	Knots/hour	1.943×10^{-2}
Centimeters/second	Meters/minute	.6
Centimeters/second	Meters/second	.01
Centimeters/second	Miles/hour	2.237×10^{-2}
Centimeters/second	Miles/minute	3.728×10^{-4}
Circular mils	Square centimeters	5.067×10^{-6}
Circular mils	Square inches	7.854×10^{-7}
Circular mils	Square millimeters	5.067×10^{-4}
Circular mils	Square mils	.7854
Coulombs	Abcoulombs	.1
Coulombs	Ampere-hours	2.778×10^{-4}
Coulombs	Faradays	1.036×10^{-5}
Coulombs	Statcoulombs	2.998×10^{9}
Cubic centimeters	Cubic feet	3.531×10^{-5}
Cubic centimeters	Cubic inches	6.102×10^{-2}
Cubic centimeters	Cubic meters	10^{-6}
Cubic centimeters	Cubic yards	1.308×10^{-6}
Cubic centimeters	Gallons (liquid)	2.642×10^{-4}
Cubic centimeters	Liters	.001
Cubic centimeters	Pints (liquid)	2.113×10^{-3}
Cubic centimeters	Quarts (liquid)	1.057×10^{-3}
Cubic feet	Bushels (dry)	.8036
Cubic feet	Cubic centimeters	2.832×10^{4}
Cubic feet	Cubic inches	1728
Cubic feet	Cubic meters	2.832×10^{-2}
Cubic feet (US)	Cubic yards	3.704×10^{-2}
Cubic feet	Gallons (liquid)	7.481
Cubic feet	Liters	28.316
Cubic feet	Pints (liquid)	59.84
Cubic feet	Quarts (liquid)	29.922
Cubic hectometers	Cubic meters	10^{6}
Cubic inches	Bushels (dry)	4.6503×10^{-4}
Cubic inches	Cubic centimeters	16.39
Cubic inches	Cubic feet	5.787×10^{-4}
Cubic inches	Cubic meters	1.639×10^{-5}
Cubic inches (US)	Cubic yards	2.143×10^{-5}
Cubic inches	Gallons	4.329×10^{-6}
Cubic inches	Liters	1.639×10^{-2}
Cubic inches	Pints (liquid)	3.463×10^{-3}
Cubic inches	Quarts (liquid)	1.732×10^{-2}
Cubic meters	Bushels (dry)	28.38
Cubic meters	Cubic centimeters	10^{6}
Cubic meters	Cubic feet	35.31
Cubic meters	Cubic inches	6.102×10^{4}
Cubic meters	Cubic yards	1.308
Cubic meters	Gallons (liquid)	264.2
Cubic meters	Liters	1000
Cubic meters	Pints (liquid)	2113
Cubic meters	Quarts (liquid)	1057
Cubic meters	Steres	1
Cubic yards	Cubic centimeters	7.646×10^{5}
Cubic yards	Cubic feet	27
Cubic yards	Cubic inches	46656
Cubic yards	Cubic meters	.7646
Cubic yards	Gallons	202.0
Cubic yards	Liters	764.6
Cubic yards	Pints (liquid)	1616
Cubic yards	Quarts (liquid)	807.9

To convert from	To	Multiply by
Decimeters	Meters	.1
Decigrams	Grams	.1
Decisteres	Cubic meters	.1
Degrees	Circumferences[a] (revolutions)	2.778×10^{-3}
Degrees	Minutes	60
Degrees	Quadrants	1.111×10^{-2}
Degrees	Radians[a]	1.745×10^{-2}
Degrees	Seconds	3600
Degrees/second	Radians/second	1.745×10^{-2}
Degrees/second	Revolutions/minute	.1667
Degrees/second	Revolutions/second	2.778×10^{-3}
Dekagrams	Grams	10
Dekameters	Meters	10
Faradays	Abcoulombs	9649
Faradays	Ampere-hours	26.81
Faradays	Coulombs	9.649×10^{4}
Faradays	Statcoulombs	2.893×10^{14}
Farads	Abfarads	10^{-9}
Farads	Microfarads	10^{6}
Farads	Statfarads	8.988×10^{11}
Feet	Centimeters	30.48
Feet	Inches	12
Feet	Kilometers	3.048×10^{-4}
Feet	Meters	.3048
Feet	Miles (nautical)	1.645×10^{-4}
Feet	Miles (statute)	1.894×10^{-4}
Feet	Mils	1.2×10^{4}
Feet	Millimeters	304.8
Feet	Yards	.3333
Feet/minute	Centimeter/second	.5080
Feet/minute	Feet/second	1.667×10^{-2}
Feet/minute	Kilometers/hour	1.829×10^{-2}
Feet/minute	Kilometers/second	3.048×10^{-4}
Feet/minute	Knots	9.868×10^{-3}
Feet/minute	Meters/minute	.3048
Feet/minute	Meters/second	5.080×10^{-3}
Feet/minute	Miles/hour	1.136×10^{-2}
Feet/minute	Miles/minute	1.894×10^{-4}
Feet/second	Centimeters/second	30.48
Feet/second	Feet/minute	60
Feet/second	Kilometers/hour	1.097
Feet/second	Kilometers/minute	1.829×10^{-2}
Feet/second	Knots/hour	.5921
Feet/second	Meters/minute	18.29
Feet/second	Meters/second	.3048
Feet/second	Miles/hour	.6818
Feet/second	Miles/minute	1.136×10^{-2}
Gallons (liquid)	Cubic centimeters	3785.
Gallons (liquid)	Cubic feet	.1337
Gallons (liquid)	Cubic inches	231
Gallons (liquid)	Cubic meters	3.785×10^{-3}
Gallons (liquid)	Cubic yards	4.951×10^{-3}
Gallons (liquid)	Liters	3.785
Gallons (liquid)	Pints (liquid)	8
Gallons (liquid)	Quarts (liquid)	4
Grains	Grams	6.480×10^{-2}
Grains	Kilograms	6.481×10^{-5}
Grains	Milligrams	64.81

[a] See notes at end of table.

To convert from	To	Multiply by
Grains	Ounces (avoirdupois)	2.286×10^{-3}
Grains	Pounds (avoirdupois)	1.429×10^{-4}
Grams	Grains	15.43
Grams	Kilograms	6.480×10^{-5}
Grams	Milligrams	64.80
Grams	Ounces (avoirdupois)	3.527×10^{-2}
Grams	Pounds (avoirdupois)	2.205×10^{-3}
Grams	Tons (long)	9.842×10^{-7}
Grams	Tons (metric)	10^{-6}
Grams	Tons (short)	1.102×10^{-6}
Hectares	Acres	2.471
Hectares	Acres	100
Hectares	Square feet	1.076×10^{5}
Hectares	Square meters	10000
Hectares	Square rods	3.954×10^{2}
Hectares	Square yards	11959.85
Hectograms	Grams	100
Hectograms	Ounces (avoirdupois)	3.527
Hectoliters	Liters	100
Hectometers	Meters	100
Hectometers	Rods	19.88
Hectometers	Yards	109.4
Hectowatts	Watts	100
Hemispheres	Spheres	.5
Hemispheres	Spherical right angles	4
Hemispheres	Steradians[b]	6.283
Henrys	Abhenrys	10^{9}
Henrys	Microhenrys	10^{6}
Henrys	Millihenrys	1000
Henrys	Stathenrys	1.113×10^{-12}
Inches	Centimeters	2.540
Inches	Feet	8.333×10^{-2}
Inches	Kilometers	2.540×10^{-5}
Inches	Meters	2.540×10^{-2}
Inches	Miles	1.578×10^{-5}
Inches	Millimeters	25.40
Inches	Mils	1000
Inches	Yards	2.778×10^{-2}
Kilograms	Grains	1.543×10^{4}
Kilograms	Grams	1000
Kilograms	Milligrams	10^{6}
Kilograms	Ounces (avoirdupois)	35.27
Kilograms	Pounds (avoirdupois)	2.205
Kilograms	Tons (long)	9.842×10^{-4}
Kilograms	Tons (metric)	.001
Kilograms	Tons (short)	1.102×10^{-3}
Kiloliters	Gallons (liquid)	264.18
Kiloliters	Liters	1000
Kilometers	Centimeters	10^{5}
Kilometers	Feet	3281
Kilometers	Inches	3.937×10^{4}
Kilometers	Meters	1000
Kilometers	Miles (nautical)	.5396
Kilometers	Miles (statute)	.6214
Kilometers	Millimeters	10^{6}
Kilometers	Mils	3.937×10^{7}
Kilometers	Yards	1094
Kilometers/hour	Centimeters/second	27.78

See notes at end of table.

To convert from	To	Multiply by
Kilometers/hour	Feet/minute	54.68
Kilometers/hour	Feet/second	.9113
Kilometers/hour	Kilometers/minute	1.667 x 10^{-2}
Kilometers/hour	Knots/hour	.5396
Kilometers/hour	Meters/minute	16.67
Kilometers/hour	Meters/second	.2778
Kilometers/hour	Miles/hour	.6214
Kilometers/hour	Miles/minute	1.036 x 10^{-2}
Kilometers/minute	Centimeters/second	1667
Kilometers/minute	Feet/minute	3281
Kilometers/minute	Feet/second	54.68
Kilometers/minute	Kilometers/hour	60
Kilometers/minute	Knots/hour	32.38
Kilometers/minute	Meters/minute	1000
Kilometers/minute	Meters/second	16.67
Kilometers/minute	Miles/hour	37.28
Kilometers/minute	Miles/minute	.6214
Kilowatt hours	Watt-hours	1000
Kilowatts	Watts	1000
Knots/hour	Centimeters/second	51.48
Knots/hour	Feet/hour	6080.20
Knots/hour	Feet/minute	101.3
Knots/hour	Feet/second	1.689
Knots/hour	Kilometers/hour	1.853
Knots/hour	Kilometers/minute	3.088 x 10^{-2}
Knots/hour	Meters/minute	30.88
Knots/hour	Meters/second	.5148
Knots/hour	Miles/hour	1.152
Knots/hour	Miles/minute	1.919 x 10^{-2}
Liters	Bushels (dry)	2.838 x 10^{-2}
Liters	Cubic centimeters	1000
Liters	Cubic feet	3.531 x 10^{-2}
Liters	Cubic inches	61.02
Liters	Cubic meters	.001
Liters	Cubic yards	1.308 x 10^{-3}
Liters	Gallons (liquid)	.2642
Liters	Pints (liquid)	2.113
Liters	Quarts (liquid)	1.057
Megacycles	Cycles	10^{6}
Megameters	Meters	10^{6}
Megohms	Abohms	.001
Megohms	Abohms	10^{15}
Megohms	Microhms	10^{12}
Megohms	Ohms	10^{6}
Megohms	Statohms	1.112 x 10^{-3}
Meters	Centimeters	100
Meters	Feet	3.281
Meters	Inches	39.37
Meters	Kilometers	.001
Meters	Megameters	10^{-6}
Meters	Miles (statute)	6.214 x 10^{-4}
Meters	Millimeters	1000
Meters	Millimicrons	10^{9}
Meters	Mils	3.937 x 10^{4}
Meters	Yards	1.094
Meters/minute	Centimeters/second	1.667
Meters/minute	Feet/minute	3.281
Meters/minute	Feet/second	5.468 x 10^{-2}
Meters/minute	Kilometers/hour	.06

To convert from	To	Multiply by
Meters/minute	Kilometers/minute	.001
Meters/minute	Knots/hour	3.238 x 10^{-2}
Meters/minute	Meters/second	1.667 x 10^{-2}
Meters/minute	Miles/hour	3.728 x 10^{-2}
Meters/minute	Miles/minute	6.214 x 10^{-4}
Meters/second	Centimeters/second	100
Meters/second	Feet/minute	196.8
Meters/second	Feet/second	3.281
Meters/second	Kilometers/hour	3.6
Meters/second	Kilometers/minute	.06
Meters/second	Knots/hour	1.943
Meters/second	Meters/minute	60
Meters/second	Miles/hour	2.237
Meters/second	Miles/minute	3.728 x 10^{-2}
Microfarads	Abfarads	10^{-15}
Microfarads	Farads	10^{-6}
Microfarads	Statfarads	8.988 x 10^{5}
Micrograms	Grams	10^{-6}
Milliograms	Milligrams	.001
Microhenrys	Abhenrys	1,000
Microhenrys	Henrys	10^{-6}
Microhenrys	Millihenrys	.001
Microhenrys	Stathenrys	1.113 x 10^{-18}
Microhms	Abohms	1000
Microhms	Megohms	10^{-12}
Microhms	Ohms	10^{-6}
Microhms	Statohms	1.113 x 10^{-18}
Microliters	Liters	10^{-6}
Micromicrofarads	Farads	10^{-12}
Microvolts	Abvolts	100
Microvolts	Millivolts	.001
Microvolts	Statvolts	3.336 x 10^{-6}
Microvolts	Volts	10^{-6}
Miles	Centimeters	1.609 x 10^{5}
Miles	Feet	5280
Miles	Inches	6.336 x 10^{4}
Miles	Kilometers	1.609
Miles	Meters	1609
Miles	Miles (nautical)	.8684
Miles	Rods	320
Miles	Yards	1760
Miles/hour	Centimeters/second	44.70
Miles/hour	Feet/minute	88
Miles/hour	Feet/second	1.467
Miles/hour	Kilometers/hour	1.609
Miles/hour	Kilometers/minute	2.682 x 10^{-2}
Miles/hour	Knots (per hour)	.8684
Miles/hour	Meters/minute	26.82
Miles/hour	Meters/second	.4470
Miles/hour	Miles/minute	1.667 x 10^{-2}
Miles/minute	Centimeters/second	2682
Miles/minute	Feet/minute	5280
Miles/minute	Feet/second	88
Miles/minute	Kilometers/hour	96.54
Miles/minute	Kilometers/minute	1.609
Miles/minute	Knots/hour	52.10
Miles/minute	Meters/minute	1609
Miles/minute	Meters/second	26.82
Miles/minute	Miles/hour	60

To convert from	To	Multiply by
Milligrams	Grains	1.543×10^{-2}
Milligrams	Grams	.001
Milligrams	Kilograms	10^{-6}
Milligrams	Ounces (avoirdupois)	3.527×10^{-5}
Milligrams	Pounds (avoirdupois)	2.205×10^{-6}
Milligrams	Tons (long)	9.842×10^{-10}
Milligrams	Tons (metric)	10^{-9}
Milligrams	Tons (short)	1.102×10^{-9}
Millihenrys	Abhenrys	10^{6}
Millihenrys	Henrys	.001
Millihenrys	Microhenrys	1000
Millihenrys	Stathenrys	1.112×10^{-15}
Milliliters	Liters	.001
Millimeters	Centimeters	.1
Millimeters	Feet	3.281×10^{-3}
Millimeters	Inches	3.937×10^{-2}
Millimeters	Kilometers	10^{-6}
Millimeters	Meters	.001
Millimeters	Miles	6.214×10^{-7}
Millimeters	Mils	39.37
Millimeters	Yards	1.094×10^{-3}
Millimicrons	Microns	.001
Millivolts	Abvolts	10^{5}
Millivolts	Microvolts	1000
Millivolts	Statvolts	3.336×10^{-6}
Millivolts	Volts	.001
Mils	Centimeters	2.540×10^{-3}
Mils	Feet	8.333×10^{-5}
Mils	Inches	.001
Mils	Kilometers	2.540×10^{-5}
Mils	Millimeters	2.540×10^{-2}
Mils	Yards	2.778×10^{-5}
Minutes (angle)	Degrees	1.667×10^{-2}
Minutes (angle)	Quadrants	1.852×10^{-4}
Minutes (angle)	Radians[a]	2.909×10^{-4}
Minutes (angle)	Revolutions[a] (circumferences)	4.630×10^{-5}
Minutes (angle)	Seconds	60
Myriagrams	Grams	10,000
Myriagrams	Kilograms	10
Myriameters	Kilometers	10
Myriameters	Meters	10,000
Myriameters	Miles	6.21370
Ohms	Abohms	10^{9}
Ohms	Megohms	10^{-6}
Ohms	Microhms	10^{6}
Ohms	Statohms	1.112×10^{-12}
Ounces (avoirdupois)	Grains	437.5
Ounces (avoirdupois)	Grams	28.35
Ounces (avoirdupois)	Kilograms	2.835×10^{-2}
Ounces (avoirdupois)	Milligrams	2.835×10^{4}
Ounces (avoirdupois)	Pounds (avoirdupois)	6.250×10^{-2}
Ounces (avoirdupois)	Tons (long)	2.790×10^{-5}
Ounces (avoirdupois)	Tons (metric)	2.835×10^{-5}
Ounces (avoirdupois)	Tons (short)	3.125×10^{-5}
Pints (liquid)	Cubic centimeters	473.2
Pints (liquid)	Cubic feet	1.671×10^{-2}
Pints (liquid)	Cubic inches	28.87
Pints (liquid)	Cubic meters	4.732×10^{-4}

[a] See notes at end of table.

To convert from	To	Multiply by
Pints (liquid)	Cubic yards	6.189×10^{-4}
Pints (liquid)	Gallons (liquid)	.125
Pounds (avoirdupois)	Grains	7000
Pounds (avoirdupois)	Grams	453.6
Pounds (avoirdupois)	Kilograms	.4536
Pounds (avoirdupois)	Milligrams	4.536×10^{5}
Pounds (avoirdupois)	Ounces (avoirdupois)	16
Pounds (avoirdupois)	Tons (long)	4.464×10^{-4}
Pounds (avoirdupois)	Tons (short)	.0005
Quadrants	Degrees	90
Quadrants	Minutes	5400
Quadrants	Radians[a]	1.571
Quadrants	Revolutions[a] (circumferences)	.25
Quadrants	Seconds	3.24×10^{5}
Quarts (liquid)	Cubic centimeters	946.4
Quarts (liquid)	Cubic feet	3.342×10^{-2}
Quarts (liquid)	Cubic inches	57.75
Quarts (liquid)	Cubic meters	9.464×10^{-4}
Quarts (liquid)	Cubic yards	1.238×10^{-3}
Quarts (liquid)	Gallons (liquid)	.25
Radians[a]	Circumferences[a]	.1591
Radians[a]	Degrees	57.30
Radians[a]	Degrees, minutes, seconds	57°, 17', 44.8"
Radians[a]	Minutes	3438
Radians[a]	Quadrants	.6366
Radians[a]	Revolutions[a]	.1591
Radians[a]	Seconds	2.063×10^{5}
Radians/second	Degrees/second	57.30
Radians/second	Revolutions/minute	9.549
Radians/second	Revolutions/second	.1592
Revolutions[a] (circumferences)	Degrees	360
Revolutions[a] (circumferences)	Minutes	2.16×10^{4}
Revolutions[a] (circumferences)	Quadrants	4
Revolutions[a] (circumferences)	Radians[a]	6.283
Revolutions[a] (circumferences)	Seconds	1.296×10^{6}
Revolutions/minute	Degrees/second	6
Revolutions/minute	Radians/second	.1047
Revolutions/minute^2	Revolutions/second^2	1.667×10^{-2}
Revolutions/second	Degrees/second	360
Revolutions/second	Radians/second	6.283
Revolutions/second	Revolutions/minute	60
Seconds (angle)	Degrees	2.778×10^{-4}
Seconds (angle)	Minutes	1.667×10^{-2}
Seconds (angle)	Quadrants	3.087×10^{-6}
Seconds (angle)	Radians[a]	4.848×10^{-6}
Seconds (angle)	Revolutions[a] (circumferences)	7.716×10^{-7}
Spheres	Hemispheres	2
Spheres	Spherical right angles	8
Spheres	Steradians	12.57
Spherical right angles	Hemispheres	.25
Spherical right angles	Spheres	.125
Spherical right angles	Steradians	1.571
Square centimeters	Circular mils	1.973×10^{5}
Square centimeters	Square decimeters	.01
Square centimeters	Square feet	1.076×10^{-3}
Square centimeters	Square inches	.1550
Square centimeters	Square kilometers	10^{-10}
Square centimeters	Square meters	.0001

[a] See notes at end of table.

To convert from	To	Multiply by
Square centimeters	Square miles	3.861×10^{-11}
Square centimeters	Square millimeters	100
Square centimeters	Square yards	1.196×10^{-4}
Square feet	Acres	2.296×10^{-5}
Square feet	Acres	9.290×10^{-4}
Square feet	Circular mils	1.833×10^{8}
Square feet	Square centimeters	929.0
Square feet	Square inches	144
Square feet	Square kilometers	9.290×10^{-8}
Square feet	Square meters	9.290×10^{-2}
Square feet	Square miles	3.587×10^{-8}
Square feet	Square millimeters	9.290×10^{4}
Square inches	Circular mils	1.273×10^{6}
Square inches	Square centimeters	6.452
Square inches	Square feet	6.944×10^{-3}
Square inches	Square kilometers	6.452×10^{-10}
Square inches	Square meters	6.452×10^{-4}
Square inches	Square millimeters	645.2
Square inches	Square yards	7.716×10^{-4}
Square kilometers	Acres	247.1
Square kilometers	Square centimeters	10^{10}
Square kilometers	Square feet	1.076×10^{7}
Square kilometers	Square inches	1.550×10^{9}
Square kilometers	Square meters	10^{6}
Square kilometers	Square miles	.3861
Square kilometers	Square millimeters	10^{12}
Square kilometers	Square yards	1.196×10^{6}
Square meters	Acres	2.471×10^{-4}
Square meters	Acres	.01
Square meters	Circular mils	1.973×10^{9}
Square meters	Square centimeters	10^{4}
Square meters	Square feet	10.76
Square meters	Square inches	1550
Square meters	Square kilometers	10^{-6}
Square meters	Square miles	3.861×10^{-7}
Square meters	Square millimeters	10^{6}
Square meters	Square yards	1.196
Square miles	Acres	640
Square miles	Square centimeters	2.590×10^{10}
Square miles	Square feet	2.788×10^{7}
Square miles	Square inches	4.015×10^{9}
Square miles	Square kilometers	2.590
Square miles	Square meters	2.590×10^{6}
Square miles	Square yards	3.098×10^{6}
Square millimeters	Circular mils	1973
Square millimeters	Square centimeters	.01
Square millimeters	Square feet	1.076×10^{-5}
Square millimeters	Square inches	1.550×10^{-3}
Square millimeters	Square kilometers	10^{-12}
Square millimeters	Square meters	10^{-6}
Square millimeters	Square miles	3.861×10^{-13}
Square millimeters	Square yards	1.196×10^{-6}
Square rods	Acres	.00625
Square rods	Square feet	272.25
Square rods	Square inches	39204
Square rods	Square meters	25.293
Square rods	Square miles	9.766×10^{-6}
Square rods	Square yards	30.25
Square yards	Acres	2.066×10^{-4}

To convert from	To	Multiply by
Square yards	Square centimeters	8361
Square yards	Square feet	9
Square yards	Square inches	1296
Square yards	Square kilometers	8.361×10^{-7}
Square yards	Square meters	.8361
Square yards	Square miles	3.228×10^{-7}
Square yards	Square millimeters	8.361×10^{-5}
Statamperes	Abamperes	3.335×10^{-11}
Statamperes	Amperes	3.335×10^{-10}
Statcoulombs	Abcoulombs	3.335×10^{-11}
Statcoulombs	Ampere-hours	9.259×10^{-14}
Statcoulombs	Coulombs	3.335×10^{-10}
Statcoulombs	Faradays	3.457×10^{-15}
Statfarads (or centimeters)	Abfarads	1.112×10^{-21}
Statfarads	Farads	1.112×10^{-12}
Statfarads	Microfarads	1.112×10^{-6}
Stathenrys	Abhenrys	8.988×10^{20}
Stathenrys	Henrys	8.988×10^{11}
Stathenrys	Microhenrys	8.988×10^{17}
Stathenrys	Millihenrys	8.988×10^{14}
Statohms	Abohms	8.988×10^{20}
Statohms	Megohms	8.988×10^{5}
Statohms	Microhms	8.988×10^{17}
Statohms	Ohms	8.988×10^{11}
Statvolts	Abvolts	2.998×10^{10}
Statvolts	Microvolts	2.998×10^{8}
Statvolts	Millivolts	2.998×10^{5}
Statvolts	Volts	299.8
Steradians	Hemispheres	.1592
Steradians	Spheres	7.958×10^{-2}
Steradians	Spherical right angles	.6366
Steres	Cubic meters	1
Steres	Liters	999.973
Tons (long)	Grams	1.016×10^{6}
Tons (long)	Kilograms	1016
Tons (long)	Milligrams	1.016×10^{9}
Tons (long)	Ounces (avoirdupois)	3.584×10^{4}
Tons (long)	Pounds (avoirdupois)	2240
Tons (long)	Tons (metric)	1.016
Tons (long)	Tons (short)	1.120
Tons (metric)	Grams	10^{6}
Tons (metric)	Kilograms	1000
Tons (metric)	Milligrams	10^{9}
Tons (metric)	Ounces (avoirdupois)	3.527×10^{4}
Tons (metric)	Pounds (avoirdupois)	2205
Tons (metric)	Tons (long)	.9842
Tons (metric)	Tons (short)	1.102
Tons (short)	Grams	9.072×10^{5}
Tons (short)	Kilograms	907.2
Tons (short)	Milligrams	9.072×10^{8}
Tons (short)	Ounces (avoirdupois)	3.2×10^{4}
Tons (short)	Pounds (avoirdupois)	2000
Tons (short)	Tons (long)	.8929
Tons (short)	Tons (metric)	.9072
Volts	Abvolts	10^{8}
Volts	Microvolts	10^{6}
Volts	Millivolts	1000
Volts	Statvolts	3.335×10^{-3}
Watts	Horsepower	.0013410

To convert from	To	Multiply by
Watts	Kilowatts	.001
Yards	Centimeters	91.44
Yards	Feet	3
Yards	Inches	36
Yards	Kilometers	9.144×10^{-4}
Yards	Meters	.9144
Yards	Miles	5.682×10^{-4}
Yards	Miles (nautical)	4.934×10^{-4}
Yards	Millimeters	914.4
Yards	Mils	3.6×10^{4}

[a] 2 π radians = 1 circumference = 360°

[b] 4 π steradians = 1 sphere.

APPENDIX III

TABLES

1. Squares, Cubes, Square Roots, and Cube Roots

No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	Area	No.
1	1	1	1.0000	1.0000	3.142	0.7854	1
2	4	8	1.4142	1.2599	6.283	3.1416	2
3	9	27	1.7321	1.4423	9.425	7.0686	3
4	16	64	2.0000	1.5874	12.566	12.5664	4
5	25	125	2.2361	1.7100	15.708	19.6350	5
6	36	216	2.4495	1.8171	18.850	28.2743	6
7	49	343	2.6458	1.9129	21.991	38.4845	7
8	64	512	2.8284	2.0000	25.133	50.2655	8
9	81	729	3.0000	2.0801	28.274	63.6173	9
10	100	1000	3.1623	2.1544	31.416	78.5398	10
11	121	1331	3.3166	2.2240	34.558	95.0332	11
12	144	1728	3.4641	2.2894	37.699	113.097	12
13	169	2197	3.6056	2.3513	40.841	132.732	13
14	196	2744	3.7417	2.4101	43.982	153.938	14
15	225	3375	3.8730	2.4662	47.124	176.715	15
16	256	4096	4.0000	2.5198	50.265	201.062	16
17	289	4913	4.1231	2.5713	53.407	226.980	17
18	324	5832	4.2426	2.6207	56.549	254.469	18
19	361	6859	4.3589	2.6684	59.690	283.529	19
20	400	8000	4.4721	2.7144	62.832	314.159	20
21	441	9261	4.5826	2.7589	65.973	346.361	21
22	484	10648	4.6904	2.8020	69.115	380.133	22
23	529	12167	4.7958	2.8439	72.257	415.476	23
24	576	13824	4.8990	2.8845	75.398	452.389	24
25	625	15625	5.0000	2.9240	78.540	490.874	25
26	676	17576	5.0990	2.9625	81.681	530.929	26
27	729	19683	5.1962	3.0000	84.823	572.555	27
28	784	21952	5.2915	3.0366	87.965	615.752	28
29	841	24389	5.3852	3.0723	91.106	660.520	29
30	900	27000	5.4772	3.1072	94.248	706.858	30
31	961	29791	5.5678	3.1414	97.389	754.768	31
32	1024	32768	5.6569	3.1748	100.531	804.248	32
33	1089	35937	5.7446	3.2075	103.673	855.299	33
34	1156	39304	5.8310	3.2396	106.814	907.920	34
35	1225	42875	5.9161	3.2711	109.956	962.113	35
36	1296	46656	6.0000	3.3019	113.097	1017.88	36
37	1369	50653	6.0828	3.3322	116.239	1075.21	37
38	1444	54872	6.1644	3.3620	119.381	1134.11	38
39	1521	59319	6.2450	3.3912	122.522	1194.59	39

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	Area	No.
40	1600	64000	6.3246	3.4200	125.66	1256.64	40
41	1681	68921	6.4031	3.4482	128.81	1320.25	41
42	1764	74088	6.4807	3.4760	131.95	1385.44	42
43	1849	79507	6.5574	3.5034	135.09	1452.20	43
44	1936	85184	6.6332	3.5303	138.23	1520.53	44
45	2025	91125	6.7082	3.5569	141.37	1590.43	45
46	2116	97336	6.7823	3.5830	144.51	1661.90	46
47	2209	103823	6.8557	3.6088	147.65	1734.94	47
48	2304	110592	6.9282	3.6342	150.80	1809.56	48
49	2401	117649	7.0000	3.6593	153.94	1885.74	49
50	2500	125000	7.0711	3.6840	157.08	1963.50	50
51	2601	132651	7.1414	3.7084	160.22	2042.82	51
52	2704	140608	7.2111	3.8325	163.36	2123.72	52
53	2809	148877	7.2801	3.7563	166.50	2206.18	53
54	2916	157464	7.3485	3.7798	169.65	2290.22	54
55	3025	166375	7.4162	3.8030	172.79	2375.83	55
56	3136	175616	7.4833	3.8259	175.93	2463.01	56
57	3249	185193	7.5498	3.8485	179.07	2551.76	57
58	3364	195112	7.6158	3.8709	182.21	2642.08	58
59	3481	205379	7.6811	3.8930	185.35	2733.97	59
60	3600	216000	7.7460	3.9149	188.50	2827.43	60
61	3721	226981	7.8102	3.9365	191.64	2922.47	61
62	3844	238328	7.8740	3.9579	194.78	3019.07	62
63	3969	250047	7.9373	3.9791	197.92	3117.25	63
64	4096	262114	8.0000	4.0000	201.06	3216.99	64
65	4225	274625	8.0623	4.0207	204.20	3318.31	65
66	4356	287496	8.1240	4.0412	207.35	3421.19	66
67	4489	300763	8.1854	4.0615	210.49	3525.65	67
68	4624	314432	8.2462	4.0817	213.63	3631.68	68
69	4761	328509	8.3066	4.1016	216.77	3739.28	69
70	4900	343000	8.3666	4.1213	219.91	3848.45	70
71	5041	357911	8.4261	4.1408	223.05	3959.19	71
72	5184	373248	8.4853	4.1602	226.19	4071.50	72
73	5329	389017	8.5440	4.1793	229.34	4185.39	73
74	5476	405224	8.6023	4.1983	232.48	4300.84	74
75	5625	421875	8.6603	4.2172	235.62	4417.86	75
76	5776	438976	8.7178	4.2358	238.76	4536.46	76
77	5929	456533	8.7750	4.2543	241.90	4656.63	77
78	6084	474552	8.8318	4.2727	245.04	4778.36	78
79	6241	493039	8.8882	4.2908	248.19	4901.67	79

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	Area	No.
80	6400	512000	8.9443	4.3089	251.33	5026.55	80
81	6561	531441	9.0000	4.3267	254.47	5153.00	81
82	6724	551368	9.0554	4.3445	257.61	5281.02	82
83	6889	571787	9.1104	4.3621	260.75	5410.61	83
84	7056	592704	9.1652	4.3795	263.89	5541.77	84
85	7225	614125	9.2195	4.3968	267.04	5674.50	85
86	7396	636056	9.2736	4.4140	270.18	5808.80	86
87	7569	658503	9.3274	4.4310	273.32	5944.68	87
88	7744	681472	9.3808	4.4480	276.46	6082.12	88
89	7921	704969	9.4340	4.4647	279.60	6221.14	89
90	8100	729000	9.4868	4.4814	282.74	6361.73	90
91	8281	753571	9.5394	4.4979	285.88	6503.88	91
92	8464	778688	9.5917	4.5144	289.03	6647.61	92
93	8649	804357	9.6437	4.5307	292.17	6792.91	93
94	8836	830584	9.6954	4.5468	295.31	6939.78	94
95	9025	857375	9.7468	4.5629	298.45	7088.22	95
96	9216	884736	9.7980	4.5789	301.59	7238.23	96
97	9409	912673	9.4889	4.5947	304.73	7389.81	97
98	9604	941192	9.8995	4.6104	307.88	7542.96	98
99	9801	970299	9.9499	4.6261	311.02	7697.69	99
100	10000	1000000	10.0000	4.6416	314.16	7853.98	100
101	10201	1030301	10.4099	4.6570	317.30	8011.85	101
102	10404	1061208	10.0995	4.6723	320.44	8171.28	102
103	10609	1092727	10.1489	4.6875	323.58	8332.29	103
104	10816	1124864	10.1980	4.7027	326.73	8494.87	104
105	11025	1157625	10.2470	4.7177	329.87	8659.01	105
106	11236	1191016	10.2956	4.7326	333.01	8824.73	106
107	11449	1225043	10.3441	4.7475	336.15	8992.02	107
108	11664	1259712	10.3923	4.7622	339.29	9160.88	108
109	11881	1295029	10.4403	4.7769	342.43	9331.32	109
110	12100	1331000	10.4881	4.7914	345.58	9503.32	110
111	12321	1367631	10.5357	4.8059	348.72	9676.89	111
112	12544	1404928	10.5830	4.8203	351.86	9852.03	112
113	12769	1442897	10.6301	4.8346	355.00	10028.7	113
114	12996	1481544	10.6771	4.8488	358.14	10207.0	114
115	13225	1520875	10.7238	4.8629	361.28	10386.9	115
116	13456	1560896	10.7703	4.8770	364.42	10568.3	116
117	13689	1601613	10.8167	4.8910	367.57	10751.3	117
118	13924	1643032	10.8628	4.9049	370.71	10935.9	118
119	14161	1685159	10.9087	4.9187	373.85	11122.0	119

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No.	Square	Cube	Square Root	Cube Root	No. ÷ Diam. Circum.	No. ÷ Diam. Area	No.
120	14400	1728000	10.9545	4.9324	376.99	11309.7	120
121	14641	1771561	11.0000	4.9461	380.13	11499.0	121
122	14884	1815848	11.0454	4.9597	383.27	11689.9	122
123	15129	1860867	11.0905	4.9732	386.42	11882.3	123
124	15376	1906624	11.1355	4.9866	389.56	12076.3	124
125	15625	1953125	11.1803	5.0000	392.70	12271.8	125
126	15876	2000376	11.2250	5.0133	395.84	12469.0	126
127	16129	2048383	11.2694	5.0265	398.98	12667.7	127
128	16384	2097152	11.3137	5.0397	402.12	12868.0	128
129	16641	2146689	11.3578	5.0528	405.27	13069.8	129
130	16900	2197000	11.4018	5.0658	408.41	13273.2	130
131	17161	2248091	11.4455	5.0788	411.55	13478.2	131
132	17424	2299968	11.4891	5.0916	414.69	13684.8	132
133	17689	2352637	11.5326	5.1045	417.83	13892.9	133
134	17956	2406104	11.5758	5.1172	420.97	14102.6	134
135	18225	2460375	11.6190	5.1299	424.12	14313.9	135
136	18496	2515456	11.6619	5.1426	427.26	14526.7	136
137	18769	2571353	11.7047	5.1551	430.40	14741.1	137
138	19044	2628072	11.7473	5.1676	433.54	14957.1	138
139	19321	2685619	11.7898	5.1801	436.68	15174.7	139
140	19600	2744000	11.8322	5.1925	439.82	15393.8	140
141	19881	2803221	11.8743	5.2048	442.96	15614.5	141
142	20164	2863288	11.9164	5.2171	446.11	15836.8	142
143	20449	2924207	11.9583	5.2293	449.25	16060.6	143
144	20736	2985984	12.0000	5.2415	452.39	16286.0	144
145	21025	3048625	12.0416	5.2536	455.53	16513.0	145
146	21316	3112136	12.0830	5.2656	458.67	16741.5	146
147	21609	3176523	12.1244	5.2776	461.81	16971.7	147
148	21904	3241792	12.1655	5.2896	464.96	17203.4	148
149	22201	3307949	12.2066	5.3015	468.10	17436.6	149
150	22500	3375000	12.2474	5.3133	471.24	17671.5	150
151	22801	3442951	12.2882	5.3251	474.38	17907.9	151
152	23104	3511808	12.3288	5.3368	477.52	18145.8	152
153	23409	3581577	12.3693	5.3485	480.66	18385.4	153
154	23716	3652264	12.4097	5.3601	483.81	18626.5	154
155	24025	3723875	12.4499	5.3717	486.95	18869.2	155
156	24336	3796416	12.4900	5.3832	490.09	19113.4	156
157	24649	3869893	12.5300	5.3947	493.23	19359.3	157
158	24964	3944312	12.5698	5.4061	496.37	19606.7	158
159	25281	4019679	12.6095	5.4175	499.51	19855.7	159

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No.	Square	Cube	Square Root	Cube Root	No. ÷ Diam. Circum.	No. ÷ Diam. Area	No.
160	25600	4096000	12.6491	5.4288	502.65	20106.2	160
161	25921	4173281	12.6886	5.4401	505.80	20358.3	161
162	26244	4251528	12.7279	5.4514	508.94	20612.0	162
163	26569	4330747	12.7671	5.4626	512.08	20867.2	163
164	26896	4410944	12.8062	5.4737	515.22	21124.1	164
165	27225	4492125	12.8452	5.4848	518.36	21382.5	165
166	27556	4574296	12.8841	5.4959	521.50	21642.4	166
167	27889	4657463	12.9228	5.5069	524.65	21904.0	167
168	28224	4741632	12.9615	5.5178	527.79	22167.1	168
169	28561	4826809	13.0000	5.5288	530.93	22431.8	169
170	28900	4913000	13.0384	5.5397	534.07	22698.0	170
171	29241	5000211	13.0767	5.5505	537.21	22965.8	171
172	29584	5088448	13.1149	5.5613	540.35	23235.2	172
173	29929	5177717	13.1529	5.5721	543.50	23506.2	173
174	30276	5268024	13.1909	5.5828	546.64	23778.7	174
175	30625	5359375	13.2288	5.5934	549.78	24052.8	175
176	30976	5451776	13.2665	5.6041	552.92	24328.5	176
177	31329	5545233	13.3041	5.6147	556.06	24605.7	177
178	31684	5639752	13.3417	5.6252	559.20	24884.6	178
179	32041	5735339	13.3791	5.6357	562.35	25164.9	179
180	32400	5832000	13.4164	5.6462	565.49	25446.9	180
181	32761	5929741	13.4536	5.6567	568.63	25730.4	181
182	33124	6028568	13.4907	5.6671	571.77	26015.5	182
183	33489	6128487	13.5277	5.6774	574.91	26302.2	183
184	33856	6229504	13.5647	5.6877	578.05	26590.4	184
185	34225	6331625	13.6015	5.6980	581.19	26880.3	185
186	34596	6434856	13.6382	5.7083	584.34	27171.6	186
187	34969	6539203	13.6748	5.7185	587.48	27464.6	187
188	35344	6644672	13.7113	5.7287	590.62	27759.1	188
189	35721	6751269	13.7477	5.7388	593.76	28055.2	189
190	36100	6859000	13.7840	5.7489	596.90	28352.9	190
191	36481	6967871	13.8203	5.7590	600.04	28652.1	191
192	36864	7077888	13.8564	5.7690	603.19	28952.9	192
193	37249	7189057	13.8924	5.7790	606.33	29255.3	193
194	37636	7301384	13.9284	5.7890	609.47	29559.2	194
195	38025	7414875	13.9642	5.7989	612.61	29864.8	195
196	38416	7529536	14.0000	5.8088	615.75	30171.9	196
197	38809	7645373	14.0357	5.8186	618.89	30480.5	197
198	39204	7762392	14.0712	5.8285	622.04	30790.7	198
199	39601	7880599	14.1067	5.8383	625.18	31102.6	199

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No.	Square	Cube	Square Root	Cube Root	No. Diam. Circum.	No. Diam. Area	No.
200	40000	8000000	14.1421	5.8480	628.32	31415.9	200
201	40401	8120601	14.1774	5.8578	631.46	31730.9	201
202	40804	8242408	14.2127	5.8675	634.60	32047.4	202
203	41209	8365427	14.2478	5.8771	637.74	32365.5	203
204	41616	8489664	14.2829	5.8868	640.89	32685.1	204
205	42025	8615125	14.3178	5.8964	644.03	33006.4	205
206	42436	8741816	14.3527	5.9059	647.17	33329.2	206
207	42849	8869743	14.3875	5.9155	650.31	33653.5	207
208	43264	8998912	14.4222	5.9250	653.45	33979.5	208
209	43681	9129329	14.4568	5.9345	656.59	34307.0	209
210	44100	9261000	14.4914	5.9439	659.73	34636.1	210
211	44521	9393931	14.5258	5.9533	662.88	34966.7	211
212	44944	9528128	14.5602	5.9627	666.02	35298.9	212
213	45369	9663597	14.5945	5.9721	669.16	35632.7	213
214	45796	9800344	14.6287	5.9814	672.30	35968.1	214
215	46225	9938375	14.6629	5.9907	675.44	36305.0	215
216	46656	10077696	14.6969	6.0000	678.58	36643.5	216
217	47089	10218313	14.7309	6.0092	681.73	36983.6	217
218	47524	10360232	14.7648	6.0185	684.87	37325.3	218
219	47961	10503459	14.7986	6.0277	688.01	37668.5	219
220	48400	10648000	14.8324	6.0368	691.15	38013.3	220
221	48841	10793861	14.8661	6.0459	694.29	38359.6	221
222	49284	10941048	14.8997	6.0550	697.43	38707.6	222
223	49729	11089567	14.9332	6.0641	700.58	39057.1	223
224	50176	11239424	14.9666	6.0732	703.72	39408.1	224
225	50625	11390625	15.0000	6.0822	706.86	39760.8	225
226	51076	11543176	15.0333	6.0912	710.00	40115.0	226
227	51529	11697083	15.0665	6.1002	713.14	40470.8	227
[illegible]	51984	11852352	15.0997	6.1091	716.28	40828.1	228
229	52441	12008989	15.1327	6.1180	719.42	41187.1	229
230	52900	12167000	15.1658	6.1269	722.57	41547.6	230
231	53361	12326391	15.1987	6.1358	725.71	41909.6	231
232	53824	12487168	15.2315	6.1446	728.85	42273.3	232
233	54289	12649337	15.2643	6.1534	731.99	42638.5	233
234	54756	12812904	15.2971	6.1622	735.13	43005.3	234
235	55225	12977875	15.3297	6.1710	738.27	43373.6	235
236	55696	13144256	15.3623	6.1797	741.42	43743.5	236
237	56169	13312053	15.3948	6.1885	744.56	44115.0	237
238	56644	13481272	15.4272	6.1972	747.70	44488.1	238
239	57121	13651919	15.4596	6.2058	750.84	44862.7	239

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No.	Square	Cube	Square Root	Cube Root	No. Diam. Circum.	No. Diam. Area	No.
240	57600	13824000	15.4919	6.2145	753.98	45238.9	240
241	58081	13997521	15.5242	6.2231	757.12	45616.7	241
242	58564	14172488	15.5563	6.2317	760.27	45996.1	242
243	59049	14348907	15.5885	6.2403	763.41	46377.0	243
244	59536	14526784	15.6205	6.2488	766.55	46759.5	244
245	60025	14706125	15.6525	6.2573	769.69	47143.5	245
246	60516	14886936	15.6844	6.2658	772.83	47529.2	246
247	61009	15069223	15.7162	6.2743	775.97	47916.4	247
248	61504	15252992	15.7480	6.2828	779.12	48305.1	248
249	62001	15438249	15.7797	6.2912	782.26	48695.5	249
250*	62500	15625000	15.8114	6.2996	785.40	49087.4	250
251	63001	15813251	15.8430	6.3080	788.54	49480.9	251
252	63504	16003008	15.8745	6.3164	791.68	49875.9	252
253	64009	16194277	15.9060	6.3247	794.82	50272.6	253
254	64516	16387064	15.9374	6.3330	797.96	50670.7	254
255	65025	16581375	15.9687	6.3413	801.11	51070.5	255
256	65536	16777216	16.0000	6.3496	804.25	51471.9	256
257	66049	16974593	16.0312	6.3579	807.39	51874.8	257
258	66564	17173512	16.0624	6.3661	810.53	52279.2	258
259	67081	17373979	16.0935	6.3743	813.67	52685.3	259
260	67600	17576000	16.1245	6.3825	816.81	53092.9	260
261	68121	17779581	16.1555	6.3907	819.96	53502.1	261
262	68644	17984728	16.1854	6.3988	823.10	53912.9	262
263	69169	18191447	16.2173	6.4070	826.24	54325.2	263
264	69696	18399744	16.2481	6.4151	829.38	54739.1	264
265	70225	18609625	16.2788	6.4232	832.52	55154.6	265
266	70756	18821096	16.3095	6.4312	835.66	55571.6	266
267	71289	19034163	16.3401	6.4393	838.81	55990.3	267
268	71824	19248832	16.3707	6.4473	841.95	56410.4	268
269	72361	19465109	16.4012	6.4553	845.09	56832.2	269
270	72900	19683000	16.4317	6.4633	848.23	57255.5	270
271	73441	19902511	16.4621	6.4713	851.37	57680.4	271
272	73984	20123648	16.4924	6.4792	854.51	58106.9	272
273	74529	20346417	16.5227	6.4872	857.66	58534.9	273
274	75076	20570824	16.5529	6.4951	860.80	58964.6	274
275	75625	20796875	16.5831	6.5030	863.94	59395.7	275
276	76176	21024576	16.6132	6.5108	867.08	59828.5	276
277	76729	21253933	16.6433	6.5187	870.22	60262.8	277
278	77284	21484952	16.6733	6.5265	873.36	60698.7	278
279	77841	21717639	16.7033	6.5343	876.50	61136.2	279

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
280	78400	21952000	16.7332	6.5421	879.65	61575.2	280
281	78961	22188041	16.7631	6.5499	882.79	62015.8	281
282	79524	22425768	16.7929	6.5577	885.93	62458.0	282
283	80089	22665187	16.8226	6.5654	889.07	62901.8	283
284	80656	22906304	16.8523	6.5731	892.21	63347.1	284
285	81225	23149125	16.8819	6.5808	895.35	63794.0	285
286	81796	23393656	16.9115	6.5885	898.50	64242.4	286
287	82369	23639903	16.9411	6.5962	901.64	64692.5	287
288	82944	23887872	16.9706	6.6039	904.78	65144.1	288
289	83521	24137569	17.0000	6.6115	907.92	65597.2	289
290	84100	24389000	17.0294	6.6191	911.06	66052.0	290
291	84681	24642171	17.0587	6.6267	914.20	66508.3	291
292	85264	24897088	17.0880	6.6343	917.35	66966.2	292
293	85849	25153757	17.1172	6.6419	920.49	67425.6	293
294	86436	25412184	17.1464	6.6494	923.63	67886.7	294
295	87025	25672375	17.1756	6.6569	926.77	68349.3	295
296	87616	25934336	17.2047	6.6644	929.91	68813.5	296
297	88209	26198073	17.2337	6.6719	933.05	69279.2	297
298	88804	26463592	17.2627	6.6794	936.19	69746.5	298
299	89401	26730899	17.2916	6.6869	939.34	70215.4	299
300	90000	27000000	17.3205	6.6943	942.48	70685.8	300
301	90601	27270901	17.3494	6.7018	945.62	71157.9	301
302	91204	27543608	17.3781	6.7092	948.76	71631.5	302
303	91809	27818127	17.4069	6.7166	951.90	72106.6	303
304	92416	28094464	17.4356	6.7240	955.04	72583.4	304
305	93025	28372625	17.4642	6.7313	958.19	73061.7	305
306	93636	28652616	17.4929	6.7387	961.33	73541.5	306
307	94249	28934443	17.5214	6.7460	964.47	74023.0	307
308	94864	29218112	17.5499	6.7533	967.61	74506.0	308
309	95481	29503629	17.5784	6.7606	970.75	74990.6	309
310	96100	29791000	17.6068	6.7679	973.89	75476.8	310
311	96721	30080231	17.6352	6.7752	977.04	75964.5	311
312	97344	30371328	17.6635	6.7824	980.18	76453.8	312
313	97969	30664297	17.6918	6.7897	983.32	76944.7	313
314	98596	30959144	17.7200	6.7969	986.46	77437.1	314
315	99225	31255875	17.7482	6.8041	989.60	77931.1	315
316	99856	31554496	17.7764	6.8113	992.74	78426.7	316
317	100489	31855013	17.8045	6.8185	995.88	78923.9	317
318	101124	32157432	17.8326	6.8256	999.03	79422.6	318
319	101761	32461759	17.8606	6.8328	1002.2	79922.9	319

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
320	102400	32768000	17.8885	6.8399	1005.3	80424.8	320
321	103041	33076161	17.9165	6.8470	1008.5	80928.2	321
322	103684	33386248	17.9444	6.8541	1011.6	81433.2	322
323	104329	33698267	17.9722	6.8612	1014.7	81939.8	323
324	104976	34012224	18.0000	6.8683	1017.9	82448.0	324
325	105625	34328125	18.0278	6.8753	1021.0	82957.7	325
326	106276	34645976	18.0555	6.8824	1024.2	83469.0	326
327	106929	34965783	18.0831	6.8894	1027.3	83981.8	327
328	107584	35287552	18.1108	6.8964	1030.4	84496.3	328
329	108241	35611289	18.1384	6.9034	1033.6	85012.3	329
330	108900	35937000	18.1659	6.9104	1036.7	85529.9	330
331	109561	36264691	18.1934	6.9174	1039.9	86049.0	331
332	110224	36594368	18.2209	6.9244	1043.0	86569.7	332
333	110889	36926037	18.2483	6.9313	1046.2	87092.0	333
334	111556	37259704	18.2757	6.9382	1049.3	87615.9	334
335	112225	37595375	18.3030	6.9451	1052.4	88141.3	335
336	112896	37933056	18.3303	6.9521	1055.6	88668.3	336
337	113569	38272753	18.3576	6.9589	1058.7	89196.9	337
338	114244	38614472	18.3848	6.9658	1061.9	89727.0	338
339	114921	38958219	18.4120	6.9727	1065.0	90258.7	339
340	115600	39304000	18.4391	6.9795	1068.1	90792.0	340
341	116281	39651821	18.4662	6.9864	1071.3	91326.9	341
342	116964	40001688	18.4932	6.9932	1074.4	91863.3	342
343	117649	40353607	18.5203	7.0000	1077.6	92401.3	343
344	118336	40707584	18.5472	7.0068	1080.7	92940.9	344
345	119025	41063625	18.5742	7.0136	1083.8	93482.0	345
346	119716	41421736	18.6011	7.0203	1087.0	94024.7	346
347	120409	41781923	18.6279	7.0271	1090.1	94569.0	347
348	121104	42144192	18.6548	7.0338	1093.3	95114.9	348
349	121801	42508549	18.6815	7.0406	1096.4	95662.3	349
350	122500	42875000	18.7083	7.0473	1099.6	96211.3	350
351	123201	43243551	18.7350	7.0540	1102.7	96761.8	351
352	123904	43614208	18.7617	7.0607	1105.8	97314.0	352
353	124609	43986977	18.7883	7.0674	1109.0	97867.7	353
354	125316	44361864	18.8149	7.0740	1112.1	98423.0	354
355	126025	44738875	18.8414	7.0807	1115.3	98979.8	355
356	126736	45118016	18.8680	7.0873	1118.4	99538.2	356
357	127449	45499293	18.8944	7.0940	1121.5	100098	357
358	128164	45882712	18.9209	7.1006	1124.7	100660	358
359	128881	46268279	18.9473	7.1072	1127.8	101223	359

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No.	Square	Cube	Square Root	Cube Root	No. Diam. Circum.	No. Diam. Area	No.
360	129600	46656000	18.9737	7.1138	1131.0	101788	360
361	130321	47045881	19.0000	7.1204	1134.1	102354	361
362	131044	47437928	19.0263	7.1269	1137.3	102922	362
363	131769	47832147	19.0526	7.1335	1140.4	103491	363
364	132496	48228544	19.0788	7.1400	1143.5	104062	364
365	133225	48627125	19.1050	7.1466	1146.7	104635	365
366	133956	49027896	19.1311	7.1531	1149.8	105209	366
367	134689	49430863	19.1572	7.1596	1153.0	105785	367
368	135424	49836032	19.1833	7.1661	1156.1	106362	368
369	136161	50243409	19.2094	7.1726	1159.2	106941	369
370	136900	50653000	19.2354	7.1791	1162.4	107521	370
371	137641	51064811	19.2614	7.1855	1165.5	108103	371
372	138384	51478848	19.2873	7.1920	1168.7	108687	372
373	139129	51895117	19.3132	7.1984	1171.8	109272	373
374	139876	52313624	19.3391	7.2048	1175.0	109858	374
375	140625	52734375	19.3649	7.2112	1178.1	110447	375
376	141376	53157376	19.3907	7.2177	1181.2	111036	376
377	142129	53582633	19.4165	7.2240	1184.4	111628	377
378	142884	54010152	19.4422	7.2304	1187.5	112221	378
379	143641	54439939	19.4679	7.2368	1190.7	112815	379
380	144400	54872000	19.4936	7.2432	1193.8	113411	380
381	145161	55306341	19.5192	7.2495	1196.9	114009	381
382	145924	55742968	19.5448	7.2558	1200.1	114608	382
383	146689	56181887	19.5704	7.2622	1203.2	115209	383
384	147456	56623104	19.5959	7.2685	1206.4	115812	384
385	148225	57066625	19.6214	7.2748	1209.5	116416	385
386	148996	57512456	19.6469	7.2811	1212.7	117021	386
387	149769	57960603	19.6723	7.2874	1215.8	117628	387
388	150544	58411072	19.6977	7.2936	1218.9	118237	388
389	151321	58863869	19.7231	7.2999	1222.1	118847	389
390	152100	59319000	19.7484	7.3061	1225.2	119459	390
391	152881	59776471	19.7737	7.3124	1228.4	120072	391
392	153664	60236288	19.7990	7.3186	1231.5	120687	392
393	154449	60698457	19.8242	7.3248	1234.6	121304	393
394	155236	61162984	19.8494	7.3310	1237.8	121922	394
395	156025	61629875	19.8746	7.3372	1240.9	122542	395
396	156816	62099136	19.8997	7.3434	1244.1	123163	396
397	157609	62570773	19.9249	7.3496	1247.2	123786	397
398	158404	63044792	19.9499	7.3558	1250.4	124410	398
399	159201	63521199	19.9750	7.3619	1253.5	125036	399

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No.	Square	Cube	Square Root	Cube Root	No. Diam. Circum.	No. Diam. Area	No.
400	160000	64000000	20.0000	7.3681	1256.6	125664	400
401	160801	64481201	20.0250	7.3742	1259.8	126293	401
402	161604	64964808	20.0499	7.3803	1262.9	126923	402
403	162409	65450827	20.0749	7.3864	1266.1	127556	403
404	163216	65939264	20.0998	7.3925	1269.2	128190	404
405	164025	66430125	20.1246	7.3986	1272.3	128825	405
406	164836	66923416	20.1494	7.4047	1275.5	129462	406
407	165649	67419143	20.1742	7.4108	1278.6	130100	407
408	166464	67917312	20.1990	7.4169	1281.8	130741	408
409	167281	68417929	20.2237	7.4229	1284.9	131382	409
410	168100	68921000	20.2485	7.4290	1288.1	132025	410
411	168921	69426531	20.2731	7.4350	1291.2	132670	411
412	169744	69934528	20.2978	7.4410	1294.3	133317	412
413	170569	70444997	20.3224	7.4470	1297.5	133965	413
414	171396	70957944	20.3470	7.4530	1300.6	134614	414
415	172225	71473375	20.3715	7.4590	1303.8	135265	415
416	173056	71991296	20.3961	7.4650	1306.9	135918	416
417	173889	72511713	20.4206	7.4710	1310.0	136572	417
418	174724	73034632	20.4450	7.4770	1313.2	137228	418
419	175561	73560059	20.4695	7.4829	1316.3	137885	419
420	176400	74088000	20.4939	7.4889	1319.5	138544	420
421	177241	74618461	20.5183	7.4948	1322.6	139205	421
422	178084	75151448	20.5426	7.5007	1325.8	139867	422
423	178929	75686967	20.5670	7.5067	1328.9	140531	423
424	179776	76225024	20.5913	7.5126	1332.0	141196	424
425	180625	76765625	20.6155	7.5185	1335.2	141863	425
426	181476	77308776	20.6398	7.5244	1338.3	142531	426
427	182329	77854483	20.6640	7.5302	1341.5	143201	427
428	183184	8402752	20.6882	7.5361	1344.6	143872	428
429	184041	78953589	20.7123	7.5420	1347.7	144545	429
430	184900	79507000	20.7364	7.5478	1350.9	145220	430
431	185761	80062991	20.7605	7.5537	1354.0	145896	431
432	186624	80621568	20.7846	7.5595	1357.2	146574	432
433	187489	81182737	20.8087	7.5654	1360.3	147254	433
434	188356	81746504	20.8327	7.5712	1363.5	147934	434
435	189225	82312875	20.8567	7.5770	1366.6	148617	435
436	190096	82881856	20.8806	7.5828	1369.7	149301	436
437	190969	83453453	20.9045	7.5886	1372.9	149987	437
438	191844	84027672	20.9284	7.5944	1376.0	150674	438
439	192721	84604519	20.9523	7.6001	1379.2	151363	439

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
440	193600	85184000	20.9762	7.6059	1382.3	152053	440
441	194481	85766121	21.0000	7.6117	1385.4	152745	441
442	195364	86350888	21.0238	7.6174	1388.6	153439	442
443	196249	86938307	21.0476	7.6232	1391.7	154134	443
444	197136	87528384	21.0713	7.6289	1394.9	154830	444
445	198025	88121125	21.0950	7.6346	1398.0	155528	445
446	198916	88716536	21.1187	7.6403	1401.2	156228	446
447	199809	89314623	21.1424	7.6460	1404.3	156930	447
448	200704	89915392	21.1660	7.6517	1407.4	157633	448
449	201601	90518849	21.1896	7.6574	1410.6	158337	449
450	202500	91125000	21.2132	7.6631	1413.7	159043	450
451	203401	91733851	21.2368	7.6688	1416.9	159751	451
452	204304	92345408	21.2603	7.6744	1420.0	160460	452
453	205209	92959677	21.2838	7.6801	1423.1	161171	453
454	206116	93576664	21.3073	7.6857	1426.3	161883	454
455	207025	94196375	21.3307	7.6914	1429.4	162597	455
456	207936	94818816	21.3542	7.6970	1432.6	163313	456
457	208849	95443993	21.3776	7.7026	1435.7	164030	457
458	209764	96071912	21.4009	7.7082	1438.9	164748	458
459	210681	96702579	21.4243	7.7138	1442.0	165468	459
460	211600	97336000	21.4476	7.7194	1445.1	166190	460
461	212521	97972181	21.4709	7.7250	1448.3	166914	461
462	213444	98611128	21.4942	7.7306	1451.4	167639	462
463	214369	99252847	21.5174	7.7362	1454.6	168365	463
464	215296	99897344	21.5407	7.7418	1457.7	169093	464
465	216225	100544625	21.5639	7.7473	1460.8	169823	465
466	217156	101194696	21.5870	7.7529	1464.0	170554	466
467	218089	101847563	21.6102	7.7584	1467.1	171287	467
468	219024	102503232	21.6333	7.7639	1470.3	172021	468
469	219961	103161709	21.6564	7.7695	1473.4	172757	469
470	220900	103823000	21.6795	7.7750	1476.5	173494	470
471	221841	104487111	21.7025	7.7805	1479.7	174234	471
472	222784	105154048	21.7256	7.7860	1482.8	174974	472
473	223729	105823817	21.7486	7.7915	1486.0	175716	473
474	224676	106496424	21.7715	7.7970	1489.1	176460	474
475	225625	107171875	21.7945	7.8025	1492.3	177205	475
476	226576	107850176	21.8174	7.8079	1495.4	177952	476
477	227529	108531333	21.8403	7.8134	1498.5	178701	477
478	228484	109215352	21.8632	7.8188	1501.7	179451	478
479	229441	109902239	21.8861	7.8243	1504.8	180203	479

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
480	230400	110592000	21.9089	7.8297	1508.0	180956	480
481	231361	111284641	21.9317	7.8352	1511.1	181711	481
482	232324	111980168	21.9545	7.8406	1514.3	182467	482
483	233289	112678587	21.9773	7.8460	1517.4	183225	483
484	234256	113379904	22.0000	7.8514	1520.5	183984	484
485	235225	114084125	22.0227	7.8568	1523.7	184745	485
486	236196	114791256	22.0454	7.8622	1526.8	185508	486
487	237169	115501303	22.0681	7.8676	1530.0	186272	487
488	238144	116214272	22.0907	7.8730	1533.1	187038	488
489	239121	116930169	22.1133	7.8784	1536.2	187805	489
490	240100	117649000	22.1359	7.8837	1539.4	188574	490
491	241081	118370771	22.1585	7.8891	1542.5	189345	491
492	242064	119095488	22.1811	7.8944	1545.7	190117	492
493	243049	119823157	22.2036	7.8998	1548.8	190890	493
494	244036	120553784	22.2261	7.9051	1551.9	191665	494
495	245025	121287375	22.2486	7.9105	1555.1	192442	495
496	246016	122023936	22.2711	7.9158	1558.2	193221	496
497	247009	122763473	22.2935	7.9211	1561.4	194000	497
498	248004	123505992	22.3159	7.9264	1564.5	194782	498
499	249001	124251499	22.3383	7.9317	1567.7	195565	499
500	250000	125000000	22.3607	7.9370	1570.8	196350	500
501	251001	125751501	22.3830	7.9423	1573.9	197136	501
502	252004	126506008	22.4054	7.9476	1577.1	197923	502
503	253009	127263527	22.4277	7.9528	1580.2	198713	503
504	254016	128024064	22.4499	7.9581	1583.4	199504	504
505	255025	128787625	22.4722	7.9634	1586.5	200296	505
506	256036	129554216	22.4944	7.9686	1589.7	201090	506
507	257049	130323843	22.5167	7.9739	1592.8	201886	507
508	258064	131096512	22.5389	7.9791	1595.9	202683	508
509	259081	131872229	22.5610	7.9843	1599.1	203482	509
510	260100	132651000	22.5832	7.9896	1602.2	204282	510
511	261121	133432831	22.6053	7.9948	1605.4	205084	511
512	262144	134217728	22.6274	8.0000	1608.5	205887	512
513	263169	135005697	22.6495	8.0052	1611.6	206692	513
514	264196	135796744	22.6716	8.0104	1614.8	207499	514
515	265225	136590875	22.6936	8.0156	1617.9	208307	515
516	266256	137388096	22.7156	8.0208	1621.1	209117	516
517	267289	138188413	22.7376	8.0260	1624.2	209928	517
518	268324	138991832	22.7596	8.0311	1627.3	210741	518
519	269361	139798359	22.7816	8.0363	1630.5	211556	519

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
520	270400	140608000	22.8035	8.0415	1633.6	212372	520
521	271441	141420761	22.8254	8.0466	1636.8	213189	521
522	272484	142236648	22.8473	8.0517	1639.9	214008	522
523	273529	143055667	22.8692	8.0569	1643.1	214829	523
524	274576	143877824	22.8910	8.0620	1646.2	215651	524
525	275625	144703125	22.9129	8.0671	1649.3	216475	525
526	276676	145531576	22.9347	8.0723	1652.5	217301	526
527	277729	146363183	22.9565	8.0774	1655.6	218128	527
528	278784	147197952	22.9783	8.0825	1658.8	218956	528
529	279841	148035889	23.0000	8.0876	1661.9	219787	529
530	280900	148877000	23.0217	8.0927	1665.0	220618	530
531	281961	149721291	23.0434	8.0978	1668.2	221452	531
532	283024	150568768	23.0651	8.1028	1671.3	222287	532
533	284089	151419437	23.0868	8.1079	1674.5	223123	533
534	285156	152273304	23.1084	8.1130	1677.6	223961	534
535	286225	153130375	23.1301	8.1180	1680.8	224801	535
536	287296	153990656	23.1517	8.1231	1683.9	225642	536
537	288369	154854153	23.1733	8.1281	1687.0	226484	537
538	289444	155720872	23.1948	8.1332	1690.2	227329	538
539	290521	156590819	23.2164	8.1382	1693.3	228175	539
540	291600	157464000	23.2379	8.1433	1696.5	229022	540
541	292681	158340421	23.2594	8.1483	1699.6	229871	541
542	293764	159220088	23.2809	8.1533	1702.7	230722	542
543	294849	160103007	23.3024	8.1583	1705.9	231574	543
544	295936	160989184	23.3238	8.1633	1709.0	232428	544
545	297025	161878625	23.3452	8.1683	1712.2	233283	545
546	298116	162771336	23.3666	8.1733	1715.3	234140	546
547	299209	163667323	23.3880	8.1783	1718.5	234998	547
548	300304	164566592	23.4094	8.1833	1721.6	235858	548
549	301401	165469149	23.4307	8.1882	1724.7	236720	549
550	302500	166375000	23.4521	8.1932	1727.9	237583	550
551	303601	167284151	23.4734	8.1982	1731.0	238448	551
552	304704	168196608	23.4947	8.2031	1734.2	239314	552
553	305809	169112377	23.5160	8.2081	1737.3	240182	553
554	306916	170031464	23.5372	8.2130	1740.4	241051	554
555	308025	170953875	23.5584	8.2180	1743.6	241922	555
556	309136	171879616	23.5797	8.2229	1746.7	242795	556
557	310249	172808693	23.6008	8.2278	1749.9	243669	557
558	311364	173741112	23.6220	8.2327	1753.0	244545	558
559	312481	174676879	23.6432	8.2377	1756.2	245422	559

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
560	313600	175616000	23.6643	8.2426	1759.3	246301	560
561	314721	176558481	23.6854	8.2475	1762.4	247181	561
562	315844	177504328	23.7065	8.2524	1765.6	248063	562
563	316969	178453547	23.7276	8.2573	1768.7	248947	563
564	318096	179406144	23.7487	8.2621	1771.9	249832	564
565	319225	180362125	23.7697	8.2670	1775.0	250719	565
566	320356	181321496	23.7908	8.2719	1778.1	251607	566
567	321489	182284263	23.8118	8.2768	1781.3	252497	567
568	322624	183250432	23.8328	8.2816	1784.4	253388	568
569	323761	184220009	23.8537	8.2865	1787.6	254281	569
570	324900	185193000	23.8747	8.2913	1790.7	255176	570
571	326041	186169411	23.8956	8.2962	1793.9	256072	571
572	327184	187149248	23.9165	8.3010	1797.0	256970	572
573	328329	188132517	23.9374	8.3059	1800.1	257869	573
574	329476	189119224	23.9583	8.3107	1803.3	258770	574
575	330625	190109375	23.9792	8.3155	1806.4	259672	575
576	331776	191102976	24.0000	8.3203	1809.6	260576	576
577	332929	192100033	24.0208	8.3251	1812.7	261482	577
578	334084	193100552	24.0416	8.3300	1815.8	262389	578
579	335241	194104539	24.0624	8.3348	1819.0	263298	579
580	336400	195112000	24.0832	8.3396	1822.1	264208	580
581	337561	196122941	24.1039	8.3443	1825.3	265120	581
582	338724	197137368	24.1247	8.3491	1828.4	266033	582
583	339889	198155287	24.1454	8.3539	1831.6	266948	583
584	341056	199176704	24.1661	8.3587	1834.7	267865	584
585	342225	200201625	24.1868	8.3634	1837.8	268783	585
586	343396	201230056	24.2074	8.3682	1841.0	269701	586
587	344569	202262003	24.2281	8.3730	1844.1	270624	587
588	345744	203297472	24.2487	8.3777	1847.3	271547	588
589	346921	204336469	24.2693	8.3825	1850.4	272471	589
590	348100	205379000	24.2899	8.3872	1853.5	273397	590
591	349281	206425071	24.3105	8.3919	1856.7	274325	591
592	350464	207474688	24.3311	8.3967	1859.8	275254	592
593	351649	208527857	24.3516	8.4014	1863.0	276184	593
594	352836	209584584	24.3721	8.4061	1866.1	277117	594
595	354025	210644875	24.3926	8.4108	1869.3	278051	595
596	355216	211708736	24.4131	8.4155	1872.4	278986	596
597	356409	212776173	24.4336	8.4202	1875.5	279923	597
598	357604	213847192	24.4540	8.4249	1878.7	280862	598
599	358801	214921799	24.4745	8.4296	1881.8	281802	599

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
600	360000	216000000	24.4949	8.4343	1885.0	282743	600
601	361201	217081801	24.5153	8.4390	1888.1	283687	601
602	362404	218167208	24.5357	8.4437	1891.2	284631	602
603	363609	219256227	24.5561	8.4484	1894.4	285578	603
604	364816	220348864	24.5764	8.4530	1897.5	286526	604
605	366025	221445125	24.5967	8.4577	1900.7	287475	605
606	367236	222545016	24.6171	8.4623	1903.8	288426	606
607	368449	223648543	24.6374	8.4670	1907.0	289379	607
608	369664	224755712	24.6577	8.4716	1910.1	290333	608
609	370881	225866529	24.6779	8.4763	1913.2	291289	609
610	372100	226981000	24.6982	8.4809	1916.4	292247	610
611	373321	228099131	24.7184	8.4856	1919.5	293206	611
612	374544	229220928	24.7386	8.4902	1922.7	294166	612
613	375769	230346397	24.7588	8.4948	1925.8	295128	613
614	376996	231475544	24.7790	8.4994	1928.9	296092	614
615	378225	232608375	24.7992	8.5040	1932.1	297057	615
616	379456	233744896	24.8193	8.5086	1935.2	298024	616
617	380689	234885113	24.8395	8.5132	1938.4	298992	617
618	381924	236029032	24.8596	8.5178	1941.5	299962	618
619	383161	237176659	24.8797	8.5224	1944.7	300934	619
620	384400	238328000	24.8998	8.5270	1947.8	301907	620
621	385641	239483061	24.9199	8.5316	1950.9	302882	621
622	386884	240641848	24.9399	8.5462	1954.1	303858	622
623	388129	241804367	24.9600	8.5408	1957.2	304836	623
624	389376	242970624	24.9800	8.5453	1960.4	305815	624
625	390625	244140625	25.0000	8.5499	1963.5	306796	625
626	391876	245314376	25.0200	8.5544	1966.6	307779	626
627	393129	246491883	25.0400	8.5590	1969.8	308763	627
628	394384	247673152	25.0599	8.5635	1972.9	309748	628
629	395641	248858189	25.0799	8.5681	1976.1	310736	629
630	396900	250047000	25.0998	8.5726	1979.2	311725	630
631	398161	251239591	25.1197	8.5772	1982.4	312715	631
632	399424	252435968	25.1396	8.5817	1985.5	313707	632
633	400689	253636137	25 1595	8.5862	1988.6	314700	633
634	401956	254840104	25.1794	8.5907	1991.8	315696	634
635	403225	256047875	25.1992	8.5952	1994.9	316692	635
636	404496	257259456	25.2190	8.5997	1998.1	317690	636
637	405769	258474853	25.2389	8.6043	2001.2	318690	637
638	407044	259694072	24 2587	8.6088	2004.3	319692	638
639	408321	260917119	25.2784	8.6132	2007.5	320695	639

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
640	409600	262144000	25.2982	8.6177	2010.6	321699	640
641	410881	263374721	25.3180	8.6222	2013.8	322705	641
642	412164	264609288	25.3377	8.6267	2016.9	323713	642
643	413449	265847707	25.3574	8.6312	2020.0	324722	643
644	414736	267089984	25.3772	8.6357	2023.2	325733	644
645	416025	268336125	25.3969	8.6401	2026.3	326745	645
646	417316	269586136	25.4165	8.6446	2029.5	327759	646
647	418609	270840023	25.4362	8.6490	2032.6	328775	647
648	419904	272097792	25.4558	8.6535	2035.8	329792	648
649	421201	273359449	25.4755	8.6579	2038.9	330810	649
650	422500	274625000	25.4951	8.6624	2042.0	331831	650
651	423801	275894451	25.5147	8.6668	2045.2	332853	651
652	425104	277167808	25.5343	8.6713	2048.3	333876	652
653	426409	278445077	25.5539	8.6757	2051.5	334901	653
654	427716	279726264	25.5734	8.6801	2054.6	335927	654
655	429025	281011375	25.5930	8.6845	2057.7	336955	655
656	430336	282300416	25.6125	8.6890	2060.9	337985	656
657	431649	283593393	25.6320	8.6934	2064.0	339016	657
658	432964	284890312	25.6515	8.6978	2067.2	340049	658
659	434281	286191179	25.6710	8.7022	2070.3	341084	659
660	435600	287496000	25.6905	8.7066	2073.5	342119	660
661	436921	288804781	25.7099	8.7110	2076.6	343157	661
662	438244	290117528	25.7294	8.7154	2079.7	344196	662
663	439569	291434247	25.7488	8.7198	2082.9	345237	663
664	440896	292754944	25.7682	8.7241	2086.0	346279	664
665	442225	294079625	25.7876	8.7285	2089.2	347323	665
666	443556	295408296	25.8070	8.7329	2092.3	348368	666
667	444889	296740963	25.8263	8.7373	2095.4	349415	667
668	446224	298077632	25.8457	8.7416	2098.6	350464	668
669	447561	299418309	25.8650	8.7460	2101.7	351514	669
670	448900	300763000	25.8844	8.7503	2104.9	352565	670
671	450241	302111711	25.9037	8.7547	2108.0	353618	671
672	451584	303464448	25.9230	8.7590	2111.2	354673	672
673	452929	304821217	25.9422	8.7634	2114.3	355730	673
674	454276	306182024	25.9615	8.7677	2117.4	356788	674
675	455625	307546875	25.9808	8.7721	2120.6	357847	675
676	456976	308915776	26.0000	8.7764	2123.7	358908	676
677	458329	310288733	26.0192	8.7807	2126.9	359971	677
678	459684	311665752	26.0384	8.7850	2130.0	361035	678
679	461041	313046839	26.0576	8.7893	2133.1	362101	679

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
680	462400	314432000	26.0768	8.7937	2136.3	363168	680
681	463761	315821241	26.0960	8.7980	2139.4	364237	681
682	465124	317214568	26.1151	8.8023	2142.6	365308	682
683	466489	318611987	26.1343	8.8066	2145.7	366380	683
684	467856	320013504	26.1534	8.8109	2148.9	367453	684
685	469225	321419125	26.1725	8.8152	2152.0	368528	685
686	470596	322828856	26.1916	8.8194	2155.1	369605	686
687	471969	324242703	26.2107	8.8237	2158.3	370684	687
688	473344	325660672	26.2298	8.8280	2161.4	371764	688
689	474721	327082769	26.2488	8.8323	2164.6	372845	689
690	476100	328509000	26.2679	8.8366	2167.7	373928	690
691	477481	329939371	26.2869	8.8408	2170.8	375013	691
692	478864	331373888	26.3059	8.8451	2174.0	376099	692
693	480249	332812557	26.3249	8.8493	2177.1	377187	693
694	481636	334255384	26.3439	8.8536	2180.3	378276	694
695	483025	335702375	26.3629	8.8578	2183.4	379367	695
696	484416	337153536	26.3818	8.8621	2186.6	380459	696
697	485809	338608873	26.4008	8.8663	2189.7	381554	697
698	487204	340068392	26.4197	8.8706	2192.8	382649	698
699	488601	341532099	26.4386	8.8748	2196.0	383746	699
700	490000	343000000	26.4575	8.8790	2199.1	384845	700
701	491401	344472101	26.4764	8.8833	2202.3	385945	701
702	492804	345948408	26.4953	8.8875	2205.4	387047	702
703	494209	347428927	26.5141	8.8917	2208.5	388151	703
704	495616	348913664	26.5330	8.8959	2211.7	389256	704
705	497025	350402625	26.5518	8.9001	2214.8	390363	705
706	498436	351895816	26.5707	8.9043	2218.0	391471	706
707	499849	353393243	26.5895	8.9085	2221.1	392580	707
708	501264	354894912	26.6083	8.9127	2224.3	393692	708
709	502681	356400829	26.6271	8.9169	2227.4	394805	709
710	504100	357911000	26.6458	8.9211	2230.5	395919	710
711	505521	359425431	26.6646	8.9253	2233.7	397035	711
712	506944	360944128	26.6833	8.9295	2236.8	398153	712
713	508369	362467097	26.7021	8.9337	2240.0	399272	713
714	509796	363994344	26.7208	8.9378	2243.1	400393	714
715	511225	365525875	26.7395	8.9420	2246.2	401515	715
716	512656	367061696	26.7582	8.9462	2249.4	402639	716
717	514089	368601813	26.7769	8.9503	2252.5	403765	717
718	515524	370146232	26.7955	8.9545	2255.7	404892	718
719	516961	371694959	26.8142	8.9587	2258.8	406020	719

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
720	518400	373248000	26.8328	8.9628	2261.9	407150	720
721	519841	374805361	26.8514	8.9670	2265.1	408282	721
722	521284	376367048	26.8701	8.9711	2268.2	409416	722
723	522729	377933067	26.8887	8.9752	2271.4	410550	723
724	524176	379503424	26.9072	8.9794	2274.5	411687	724
725	525625	381078125	26.9258	8.9835	2277.7	412825	725
726	527076	382657176	26.9444	8.9876	2280.8	413965	726
727	528529	384240583	26.9629	8.9918	2283.9	415106	727
728	529984	385828352	26.9815	8.9959	2287.1	416248	728
729	531441	387420489	27.0000	9.0000	2290.2	417393	729
730	532900	389017000	27.0185	9.0041	2293.4	418539	730
731	534361	390617891	27.0370	9.0082	2296.5	419686	731
732	535824	392223168	27.0555	9.0123	2299.7	420835	732
733	537289	393832837	27.0740	9.0164	2302.8	421986	733
734	538756	395446904	27.0924	9.0205	2305.9	423138	734
735	540225	397065375	27.1109	9.0246	2309.1	424293	735
736	541696	398688256	27.1293	9.0287	2312.2	425448	736
737	543169	400315553	27.1477	9.0328	2315.4	426604	737
738	544644	401947272	27.1662	9.0369	2318.5	427762	738
739	546121	403583419	27.1846	9.0410	2321.6	428922	739
740	547600	405224000	27.2029	9.0450	2324.8	430084	740
741	549081	406869021	27.2213	9.0491	2327.9	431247	741
742	550564	408518488	27.2397	9.0532	2331.1	432412	742
743	552049	410172407	27.2580	9.0572	2334.2	433578	743
744	553536	411830784	27.2764	9.0613	2337.3	434746	744
745	555025	413493625	27.2947	9.0654	2340.5	435916	745
746	556516	415160936	27.3130	9.0694	2343.6	437087	746
747	558009	416832723	27.3313	9.0735	2346.8	438259	747
748	559504	418508992	27.3496	9.0775	2349.9	439433	748
749	561001	420189749	27.3679	9.0816	2353.1	440609	749
750	562500	421875000	27.3861	9.0856	2356.2	441786	750
751	564001	423564751	27.4044	9.0896	2359.3	442965	751
752	565504	425259008	27.4226	9.0937	2362.5	444146	752
753	567009	426957777	27.4408	9.0977	2365.6	445328	753
754	568516	428661064	27.4591	9.1017	2368.8	446511	754
755	570025	430368875	27.4773	9.1057	2371.9	447697	755
756	571536	432081216	27.4955	9.1098	2375.0	448883	756
757	573049	433798093	27.5136	9.1138	2378.2	450072	757
758	574564	435519512	27.5318	9.1178	2381.3	451262	758
759	576081	437245479	27.5500	9.1218	2384.5	452453	759

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
760	577600	438976000	27.5681	9.1258	2387.6	453646	760
761	579121	440711081	27.5862	9.1298	2390.8	454841	761
762	580644	442450728	27.6043	9.1338	2393.9	456037	762
763	582169	444194947	27.6225	9.1378	2397.0	457234	763
764	583696	445943744	27.6405	9.1418	2400.2	458434	764
765	585225	447697125	27.6586	9.1458	2403.3	459635	765
766	586756	449455096	27.6767	9.1498	2406.5	460837	766
767	588289	451217663	27.6948	9.1537	2409.6	462042	767
768	589824	452984832	27.7128	9.1577	2412.7	463247	768
769	591361	454756609	27.7308	9.1617	2415.9	464454	769
770	592900	456533000	27.7489	9.1657	2419.0	465663	770
771	594441	458314011	27.7669	9.1696	2422.2	466873	771
772	595984	460099648	27.7849	9.1736	2425.3	468085	772
773	597529	461889917	27.8029	9.1775	2428.5	469298	773
774	599076	463684824	27.8209	9.1815	2431.6	470513	774
775	600625	465484375	27.8388	9.1855	2434.7	471730	775
776	602176	467288576	27.8568	9.1894	2437.9	472948	776
777	603729	469097433	27.8747	9.1933	2441.0	474168	777
778	605284	470910952	27.8927	9.1973	2444.2	475389	778
779	606841	472729139	27.9106	9.2012	2447.3	476612	779
780	608400	474552000	27.9285	9.2052	2450.4	477836	780
781	609961	476379541	27.9464	9.2091	2453.6	479062	781
782	611524	478211768	27.9643	9.2130	2456.7	480290	782
783	613089	480048687	27.9821	9.2170	2459.9	481519	783
784	614656	481890304	28.0000	9.2209	2463.0	482750	784
785	616225	483736625	28.0179	9.2248	2466.2	483982	785
786	617796	485587656	28.0357	9.2287	2469.3	485216	786
787	619369	487443403	28.0535	9.2326	2472.4	486451	787
788	620944	489303872	28.0713	9.2365	2475.6	487688	788
789	622521	491169069	28.0891	9.2404	2478.7	488927	789
790	624100	493039000	28.1069	9.2443	2481.9	490167	790
791	625681	494913671	28.1247	9.2482	2485.0	491409	791
792	627264	496793088	28.1425	9.2521	2488.1	492652	792
793	628849	498677257	28.1603	9.2560	2491.3	493897	793
794	630436	500566184	28.1780	9.2599	2494.4	495143	794
795	632025	502459875	28.1957	9.2638	2497.6	496391	795
796	633616	504358336	28.2135	9.2677	2500.7	497641	796
797	635209	506261573	28.2312	9.2716	2503.8	498892	797
798	636804	508169592	28.2489	9.2754	2507.0	500145	798
799	638401	510082399	28.2666	9.2793	2510.1	501399	799

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
800	640000	512000000	28.2843	9.2832	2513.3	502655	800
801	641601	513922401	28.3019	9.2870	2516.4	503912	801
802	643204	515849608	28.3196	9.2909	2519.6	505171	802
803	644809	517781627	28.3373	9.2948	2522.7	506432	803
804	646416	519718464	28.3549	9.2986	2525.8	507694	804
805	648025	521660125	28.3725	9.3025	2529.0	508958	805
806	649636	523606616	28.3901	9.3063	2532.1	510223	806
807	651249	525557943	28.4077	9.3102	2535.3	511490	807
808	652864	527514112	28.4253	9.3140	2538.4	512758	808
809	654481	529475129	28.4429	9.3179	2541.5	514028	809
810	656100	531441000	28.4605	9.3217	2544.7	515300	810
811	657721	533411731	28.4781	9.3255	2547.8	516573	811
812	659344	535387328	28.4956	9.3294	2551.0	517848	812
813	660969	537367797	28.5132	9.3332	2554.1	519124	813
814	662596	539353144	28.5307	9.3370	2557.3	520402	814
815	664225	541343375	28.5482	9.3408	2560.4	521681	815
816	665856	543338496	28.5657	9.3447	2563.5	522962	816
817	667489	545338513	28.5832	9.3485	2566.7	524245	817
818	669124	547343432	28.6007	9.3523	2569.8	525529	818
819	670761	549353259	28.6182	9.3561	2573.0	526814	819
820	672400	551368000	28.6356	9.3599	2576.1	528102	820
821	674041	553387661	28.6531	9.3637	2579.2	529391	821
822	675684	555412248	28.6705	9.3675	2582.4	530681	822
823	677329	557441767	28.6880	9.3713	2585.5	531973	823
824	678976	559476224	28.7054	9.3751	2588.7	533267	824
825	680625	561515625	28.7228	9.3789	2591.8	534562	825
826	682276	563559976	28.7402	9.3827	2595.0	535858	826
827	683929	565609283	28.7576	9.3865	2598.1	537157	827
828	685584	567663552	28.7750	9.3902	2601.2	538456	828
829	687241	569722789	28.7924	9.3940	2604.4	539758	829
830	688900	571787000	28.8097	9.3978	2607.5	541061	830
831	690561	573856191	28.8271	9.4016	2610.7	542365	831
832	692224	575930368	28.8444	9.4053	2613.8	543671	832
833	693889	578009537	28.8617	9.4091	2616.9	544979	833
834	695556	580093704	28.8791	9.4129	2620.1	546288	834
835	697225	582182875	28.8964	9.4166	2623.2	547599	835
836	698896	584277056	28.9137	9.4204	2626.4	548912	836
837	700569	586376253	28.9310	9.4241	2629.5	550226	837
838	702244	588480472	28.9482	9.4279	2632.7	551541	838
839	703921	590589719	28.9655	9.4316	2635.8	552858	839

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
840	705600	592704000	28.9828	9.4354	2638.9	554177	840
841	707281	594823321	29.0000	9.4391	2642.1	555497	841
842	708964	596947688	29.0172	9.4429	2645.2	556819	842
843	710649	599077107	29.0345	9.4466	2648.4	558142	843
844	712336	601211584	29.0517	9.4503	2651.5	559467	844
845	714025	603351125	29.0689	9.4541	2654.6	560794	845
846	715716	605495736	29.0861	9.4578	2657.8	562122	846
847	717409	607645423	29.1033	9.4615	2660.9	563452	847
848	719104	609800192	29.1204	9.4652	2664.1	564783	848
849	720801	611960049	29.1376	9.4690	2667.2	566116	849
850	722500	614125000	29.1548	9.4727	2670.4	567450	850
851	724201	616295051	29.1719	9.4764	2673.5	568786	851
852	725904	618470208	29.1890	9.4801	2676.6	570124	852
853	727609	620650477	29.2062	9.4838	2679.8	571463	853
854	729316	622835864	29.2233	9.4875	2682.9	572803	854
855	731025	625026375	29.2404	9.4912	2686.1	574146	855
856	732736	627222016	29.2575	9.4949	2689.2	575490	856
857	734449	629422793	29.2746	9.4986	2692.3	576835	857
858	736164	631628712	29.2916	9.5023	2695.5	578182	858
859	737881	633839779	29.3087	9.5060	2698.6	579530	859
860	739600	636056000	29.3258	9.5097	2701.8	580880	860
861	741321	638277381	29.3428	9.5134	2704.9	582232	861
862	743044	640503928	29.3598	9.5171	2708.1	583585	862
863	744769	642735647	29.3769	9.5207	2711.2	584940	863
864	746496	644972544	29.3939	9.5244	2714.3	586297	864
865	748225	647214625	29.4109	9.5281	2717.5	587655	865
866	749956	649461896	29.4279	9.5317	2720.6	589014	866
867	751689	651714363	29.4449	9.5354	2723.8	590375	867
868	753424	653972032	29.4618	9.5391	2726.9	591738	868
869	755161	656234909	29.4788	9.5427	2730.0	593102	869
870	756900	658503000	29.4958	9.5464	2733.2	594468	870
871	758641	660776311	29.5127	9.5501	2736.3	595835	871
872	760384	663054848	29.5296	9.5537	2739.5	597204	872
873	762129	665338617	29.5466	9.5574	2742.6	598575	873
874	763876	667627624	29.5635	9.5610	2745.8	599947	874
875	765625	669921875	29.5804	9.5647	2748.9	601320	875
876	767376	672221376	29.5973	9.5683	2752.0	602696	876
877	769129	574526133	29.6142	9.5719	2755.2	604073	877
878	770884	676836152	29.6311	9.5756	2758.3	605451	878
879	772641	679151439	29.6479	9.5792	2761.5	606831	879

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
880	774400	681472000	29.6648	9.5828	2764.6	608212	880
881	776161	683797841	29.6816	9.5865	2767.7	609595	881
882	777924	686128968	29.6985	9.5901	2770.9	610980	882
883	779689	688465387	29.7153	9.5937	2774.0	612366	883
884	781456	690807104	29.7321	9.5973	2777.2	613754	884
885	783225	693154125	29.7489	9.6010	2780.3	615143	885
886	784996	695506456	29.7658	9.6046	2783.5	616534	886
887	786769	697864103	29.7825	9.6082	2786.6	617927	887
888	788544	700227072	29.7993	9.6118	2789.7	619321	888
889	790321	702595369	29.8161	9.6154	2792.9	620717	889
890	792100	704969000	29.8329	9.6190	2796.0	622114	890
891	793881	707347971	29.8496	9.6226	2799.2	623513	891
892	795664	709732288	29.8664	9.6262	2802.3	624913	892
893	797449	712121957	29.8831	9.6298	2805.4	626315	893
894	799236	714516984	29.8998	9.6334	2808.6	627718	894
895	801025	716917375	29.9166	9.6370	2811.7	629124	895
896	802816	719323136	29.9333	9.6406	2814.9	630530	896
897	804609	721734273	29.9500	9.6442	2818.0	631938	897
898	806404	724150792	29.9666	9.6477	2821.2	633348	898
899	808201	726572699	29.9833	9.6513	2824.3	634760	899
900	810000	729000000	30.0000	9.6549	2827.4	636173	900
901	811801	731432701	30.0167	9.6585	2830.6	637587	901
902	813604	733870808	30.0333	9.6620	2833.7	639003	902
903	815409	736314327	30.0500	9.6656	2836.9	640421	903
904	817216	738763264	30.0666	9.6692	2840.0	641840	904
905	819025	741217625	30.0832	9.6727	2843.1	643261	905
906	820836	743677416	30.0998	9.6763	2846.3	644683	906
907	822649	746142643	30.1164	9.6799	2849.4	646107	907
908	824464	748613312	30.1330	9.6834	2852.6	647533	908
909	826281	751089429	30.1496	9.6870	2855.7	648960	909
910	828100	753571000	30.1662	9.6905	2858.8	650388	910
911	829921	756058031	30.1828	9.6941	2862.0	651818	911
912	831744	758550528	30.1993	9.6976	2865.1	653250	912
913	833569	761048497	30.2159	9.7012	2868.3	654684	913
914	835396	763551944	30.2324	9.7017	2871.4	656118	914
915	837225	766060875	30.2490	9.7082	2874.6	657555	915
916	839056	768575296	30.2655	9.7118	2877.7	658993	916
917	840889	771095213	30.2820	9.7153	2880.8	660433	917
918	842724	773620632	30.2985	9.7188	2884.0	661874	918
919	844561	776151559	30.3150	9.7224	2887.1	663317	919

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No.	Square	Cube	Square Root	Cube Root	No. = Diam. Circum.	No. = Diam. Area	No.
920	846400	778688000	30.3315	9.7259	2890.3	664761	920
921	848241	781229961	30.3480	9.7294	2893.4	666207	921
922	850084	783777448	30.3645	9.7329	2896.5	667654	922
923	851929	786330467	30.3809	9.7364	2899.7	669103	923
924	853776	788889024	30.3974	9.7400	2902.8	670554	924
925	855625	791453125	30.4138	9.7435	2906.0	672006	925
926	857476	794022776	30.4302	9.7470	2909.1	673460	926
927	859329	796597983	30.4467	9.7505	2912.3	674915	927
928	861184	799178752	30.4631	9.7540	2915.4	676372	928
929	863041	801765089	30.4795	9.7575	2918.5	677831	929
930	864900	804357000	30.4959	9.7610	2921.7	679291	930
931	866761	806954491	30.5123	9.7645	2924.8	680752	931
932	868624	809557568	30.5287	9.7680	2928.0	682216	932
933	870489	812166237	30.5450	9.7715	2931.1	683680	933
934	872356	814780504	30.5614	9.7750	2934.2	685147	934
935	874225	817400375	30.5778	9.7785	2937.4	686615	935
936	876096	820025856	30.5941	9.7819	2940.5	688084	936
937	877969	822656953	30.6105	9.7854	2943.7	689555	937
938	879844	825293672	30.6268	9.7889	2946.8	691028	938
939	881721	827936019	30.6431	9.7924	2950.0	692502	939
940	883600	830584000	30.6594	9.7959	2953.1	693978	940
941	885481	833237621	30.6757	9.7993	2956.2	695455	941
942	887364	835896888	30.6920	9.8028	2959.4	696934	942
943	889249	838561807	30.7083	9.8063	2962.5	698415	943
944	891136	841232384	30.7246	9.8097	2965.7	699897	944
945	893025	843908625	30.7409	9.8132	2968.8	701380	945
946	894916	846590536	30.7571	9.8167	2971.9	702865	946
947	896809	849278123	30.7734	9.8201	2975.1	704352	947
948	898704	851971392	30.7896	9.8236	2978.2	705840	948
949	900601	854670349	30.8058	9.8270	2981.4	707330	949
950	902500	857375000	30.8221	9.8305	2984.5	708822	950
951	904401	860085351	30.8383	9.8339	2987.7	710315	951
952	906304	862801408	30.8545	9.8374	2990.8	711809	952
953	908209	865523177	30.8707	9.8408	2993.9	713306	953
954	910116	868250664	30.8869	9.8443	2997.1	714803	954
955	912025	870983875	30.9031	9.8477	3000.2	716303	955
956	913936	873722816	30.9192	9.8511	3003.4	717804	956
957	915849	876467493	30.9354	9.8546	3006.5	719306	957
958	917764	879217912	30.9516	9.8580	3009.6	720810	958
959	919681	881974079	30.9677	9.8614	3012.8	722316	959
960	921600	884736000	30.9839	9.8648	3015.9	723823	960
961	923521	887503681	31.0000	9.8683	3019.1	725332	961
962	925444	890277128	31.0161	9.8717	3022.2	726842	962
963	927369	893056347	31.0322	9.8751	3025.4	728354	963
964	929296	895841344	31.0483	9.8785	3028.5	729867	964
965	931225	898632125	31.0644	9.8819	3031.6	731382	965
966	933156	901428696	31.0805	9.8854	3034.8	732899	966
967	935089	904231063	31.0966	9.8888	3037.9	734417	967
968	937024	907039232	31.1127	9.8922	3041.1	735937	968
969	938961	909853209	31.1288	9.8956	3044.2	737458	969
970	940900	912673000	31.1448	9.8990	3047.3	738981	970
971	942841	915498611	31.1609	9.9024	3050.5	740506	971
972	944784	918330048	31.1769	9.9058	3053.6	742032	972
973	946729	921167317	31.1929	9.9092	3056.8	743559	973
974	948676	924010424	31.2090	9.9126	3059.9	745088	974
975	950625	926859375	31.2250	9.9160	3063.1	746619	975
976	952576	929714176	31.2410	9.9194	3066.2	748151	976
977	954529	932574833	31.2570	9.9227	3069.3	749685	977
978	956484	935441352	31.2730	9.9261	3072.5	751221	978
979	958441	938313739	31.2890	9.9295	3075.6	752758	979
980	960400	941192000	31.3050	9.9329	3078.8	754296	980
981	962361	944076141	31.3209	9.9363	3081.9	755837	981
982	964324	946966168	31.3369	9.9396	3085.0	757378	982
983	966289	949862087	31.3528	9.9430	3088.2	758922	983
984	968256	952763904	31.3688	9.9464	3091.3	760466	984
985	970225	955671625	31.3847	9.9497	3094.5	762013	985
986	972196	958585256	31.4006	9.9531	3097.6	763561	986
987	974169	961504803	31.4166	9.9565	3100.8	765111	987
988	976144	964430272	31.4325	9.9598	3103.9	766662	988
989	978121	967361669	31.4484	9.9632	3107.0	768214	989
990	980100	970299000	31.4643	9.9666	3110.2	769769	990
991	982081	973242271	31.4802	9.9699	3113.3	771325	991
992	984064	976191488	31.4960	9.9733	3116.5	772882	992
993	986049	979146657	31.5119	9.9766	3119.6	774441	993
994	988036	982107784	31.5278	9.9800	3122.7	776002	994
995	990025	985074875	31.5436	9.9833	3125.9	777564	995
996	992016	988047936	31.5595	9.9866	3129.0	779128	996
997	994009	991026973	31.5753	9.9900	3132.2	780693	997
998	996004	994011992	31.5911	9.9933	3135.3	782260	998
999	998001	997002999	31.6070	9.9967	3138.5	783828	999

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2. Common Logarithms

N	0	1	2	3	4	5	6	7	8	9	s.d.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4.2
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	3.8
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3.5
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3.2
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3.0
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	2.8
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	2.6
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2.5
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2.4
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2.2
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2.1
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2.0
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	1.9
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	1.8
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	1.8
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	1.7
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	1.6
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	1.6
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	1.5
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1.5
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1.4
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1.4
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1.3
33	5185	5198	5211	5224	5237	5250	5263	6276	5289	5302	1.3
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1.3
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1.2
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1.2
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1.2
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1.1
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1.1
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1.1
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1.0
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1.0
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1.0
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1.0
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1.0
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	.9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	.9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	.9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	.9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	.9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	.8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	.8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	.8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	.8

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N	0	1	2	3	4	5	6	7	8	9	s.d.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	.8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	.8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	.8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	.7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	.7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	.7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	.7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	.7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	.7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	.7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	.7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	.7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	.6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	.6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	.6
70	8451	8457	8563	8470	8476	8482	8488	8494	8500	8506	.6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	.6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	.6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	.6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	.6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	.6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	.6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	.6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	.6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	.5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	.5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	.5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	.5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	.5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	.5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	.5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	.5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	.5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	.5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	.5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	.5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	.5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	.5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	.5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	.5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	.5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	.5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	.4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	.4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	.4

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′	0°		1°		2°		3°		4°		′
	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	
0	.00000	One.	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	60
1	.00029	One.	.01774	.99984	.03519	.99938	.05263	.99861	.07005	.99754	59
2	.00058	One.	.01803	.99984	.03548	.99937	.05292	.99860	.07034	.99752	58
3	.00087	One.	.01832	.99983	.03577	.99936	.05321	.99858	.07063	.99750	57
4	.00116	One.	.01862	.99983	.03606	.99935	.05350	.99857	.07092	.99748	56
5	.00145	One.	.01891	.99982	.03635	.99934	.05379	.99855	.07121	.99746	55
6	.00175	One.	.01920	.99982	.03664	.99933	.05408	.99854	.07150	.99744	54
7	.00204	One.	.01949	.99981	.03693	.99932	.05437	.99852	.07179	.99742	53
8	.00233	One.	.01978	.99980	.03723	.99931	.05466	.99851	.07208	.99740	52
9	.00262	One.	.02007	.99980	.03752	.99930	.05495	.99849	.07237	.99738	51
10	.00291	One.	.02036	.99979	.03781	.99929	.05524	.99847	.07266	.99736	50
11	.00320	.99999	.02065	.99979	.03810	.99927	.05553	.99846	.07295	.99734	49
12	.00349	.99999	.02094	.99978	.03839	.99926	.05582	.99844	.07324	.99731	48
13	.00378	.99999	.02123	.99977	.03868	.99925	.05611	.99842	.07353	.99729	47
14	.00407	.99999	.02152	.99977	.03897	.99924	.05640	.99841	.07382	.99727	46
15	.00436	.99999	.02181	.99976	.03926	.99923	.05669	.99839	.07411	.99725	45
16	.00465	.99999	.02211	.99976	.03955	.99922	.05698	.99838	.07440	.99723	44
17	.00495	.99999	.02240	.99975	.03984	.99921	.05727	.99836	.07469	.99721	43
18	.00524	.99999	.02269	.99974	.04013	.99919	.05756	.99834	.07498	.99719	42
19	.00553	.99998	.02298	.99974	.04042	.99918	.05785	.99833	.07527	.99716	41
20	.00582	.99998	.02327	.99973	.04071	.99917	.05814	.99831	.07556	.99714	40
21	.00611	.99998	.02356	.99972	.04100	.99916	.05844	.99829	.07585	.99712	39
22	.00640	.99998	.02385	.99972	.04129	.99915	.05873	.99827	.07614	.99710	38
23	.00669	.99998	.02414	.99971	.04159	.99913	.05902	.99826	.07643	.99708	37
24	.00698	.99998	.02443	.99970	.04188	.99912	.05931	.99824	.07672	.99705	36
25	.00727	.99997	.02472	.99969	.04217	.99911	.05960	.99822	.07701	.99703	35
26	.00756	.99997	.02501	.99969	.04246	.99910	.05989	.99821	.07730	.99701	34
27	.00785	.99997	.02530	.99968	.04275	.99909	.06018	.99819	.07759	.99699	33
28	.00814	.99997	.02560	.99967	.04304	.99907	.06047	.99817	.07788	.99696	32
29	.00844	.99996	.02589	.99966	.04333	.99906	.06076	.99815	.07817	.99694	31
30	.00873	.99996	.02618	.99966	.04362	.99905	.06105	.99813	.07846	.99692	30
31	.00902	.99996	.02647	.99965	.04391	.99904	.06134	.99812	.07875	.99689	29
32	.00931	.99996	.02676	.99964	.04420	.99902	.06163	.99810	.07904	.99687	28
33	.00960	.99995	.02705	.99963	.04449	.99901	.06192	.99808	.07933	.99685	27
34	.00989	.99995	.02734	.99963	.04478	.99900	.06221	.99806	.07962	.99683	26
35	.01018	.99995	.02763	.99962	.04507	.99898	.06250	.99804	.07991	.99680	25
36	.01047	.99995	.02792	.99961	.04536	.99897	.06279	.99803	.08020	.99678	24
37	.01076	.99994	.02821	.99960	.04565	.99896	.06308	.99801	.08049	.99676	23
38	.01105	.99994	.02850	.99959	.04594	.99894	.06337	.99799	.08078	.99673	22
39	.01134	.99994	.02879	.99959	.04623	.99893	.06366	.99797	.08107	.99671	21
40	.01164	.99993	.02908	.99958	.04653	.99892	.06395	.99795	.08136	.99668	20
41	.01193	.99993	.02938	.99957	.04682	.99890	.06424	.99793	.08165	.99666	19
42	.01222	.99993	.02967	.99956	.04711	.99889	.06453	.99792	.08194	.99664	18
43	.01251	.99992	.02996	.99955	.04740	.99888	.06482	.99790	.08223	.99661	17
44	.01280	.99992	.03025	.99954	.04769	.99886	.06511	.99788	.08252	.99659	16
45	.01309	.99991	.03054	.99953	.04798	.99885	.06540	.99786	.08281	.99657	15
46	.01338	.99991	.03083	.99952	.04827	.99883	.06569	.99784	.08310	.99654	14
47	.01367	.99991	.03112	.99952	.04856	.99882	.06598	.99782	.08339	.99652	13
48	.01396	.99990	.03141	.99951	.04885	.99881	.06627	.99780	.08368	.99649	12
49	.01425	.99990	.03170	.99950	.04914	.99879	.06656	.99778	.08397	.99647	11
50	.01454	.99989	.03199	.99949	.04943	.99878	.06685	.99776	.08426	.99644	10
51	.01483	.99989	.03228	.99948	.04972	.99876	.06714	.99774	.08455	.99642	9
52	.01513	.99989	.03257	.99947	.05001	.99875	.06743	.99772	.08484	.99639	8
53	.01542	.99988	.03286	.99946	.05030	.99873	.06773	.99770	.08513	.99637	7
54	.01571	.99988	.03316	.99945	.05059	.99872	.06802	.99768	.08542	.99635	6
55	.01600	.99987	.03345	.99944	.05088	.99870	.06831	.99766	.08571	.99632	5
56	.01629	.99987	.03374	.99943	.05117	.99869	.06860	.99764	.08600	.99630	4
57	.01658	.99986	.03403	.99942	.05146	.99867	.06889	.99762	.08629	.99627	3
58	.01687	.99986	.03432	.99941	.05175	.99866	.06918	.99760	.08658	.99625	2
59	.01716	.99985	.03461	.99940	.05205	.99864	.06947	.99758	.08687	.99622	1
60	.01745	.99985	.03490	.99939	.05234	.99863	.06976	.99756	.08716	.99619	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	89°		88°		87°		86°		85°		

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′	5° Sine	5° Cosin	6° Sine	6° Cosin	7° Sine	7° Cosin	8° Sine	8° Cosin	9° Sine	9° Cosin	′
0	.08716	.99619	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	60
1	.08745	.99617	.10482	.99449	.12216	.99251	.13946	.99023	.15672	.98764	59
2	.08774	.99614	.10511	.99446	.12245	.99248	.13975	.99019	.15701	.98760	58
3	.08803	.99612	.10540	.99443	.12274	.99244	.14004	.99015	.15730	.98755	57
4	.08831	.99609	.10569	.99440	.12302	.99240	.14033	.99011	.15758	.98751	56
5	.08860	.99607	.10597	.99437	.12331	.99237	.14061	.99006	.15787	.98746	55
6	.08889	.99604	.10626	.99434	.12360	.99233	.14090	.99002	.15816	.98741	54
7	.08918	.99602	.10655	.99431	.12389	.99230	.14119	.98998	.15845	.98737	53
8	.08947	.99599	.10684	.99428	.12418	.99226	.14148	.98994	.15873	.98732	52
9	.08976	.99596	.10713	.99424	.12447	.99222	.14177	.98990	.15902	.98728	51
10	.09005	.99594	.10742	.99421	.12476	.99219	.14205	.98986	.15931	.98723	50
11	.09034	.99591	.10771	.99418	.12504	.99215	.14234	.98982	.15959	.98718	49
12	.09063	.99588	.10800	.99415	.12533	.99211	.14263	.98978	.15988	.98714	48
13	.09092	.99586	.10829	.99412	.12562	.99208	.14292	.98973	.16017	.98709	47
14	.09121	.99583	.10858	.99409	.12591	.99204	.14320	.98969	.16046	.98704	46
15	.09150	.99580	.10887	.99406	.12620	.99200	.14349	.98965	.16074	.98700	45
16	.09179	.99578	.10916	.99402	.12649	.99197	.14378	.98961	.16103	.98695	44
17	.09208	.99575	.10945	.99399	.12678	.99193	.14407	.98957	.16132	.98690	43
18	.09237	.99572	.10973	.99396	.12706	.99189	.14436	.98953	.16160	.98686	42
19	.09266	.99570	.11002	.99393	.12735	.99186	.14464	.98948	.16189	.98681	41
20	.09295	.99567	.11031	.99390	.12764	.99182	.14493	.98944	.16218	.98676	40
21	.09324	.99564	.11060	.99386	.12793	.99178	.14522	.98940	.16246	.98671	39
22	.09353	.99562	.11089	.99383	.12822	.99175	.14551	.98936	.16275	.98667	38
23	.09382	.99559	.11118	.99380	.12851	.99171	.14580	.98931	.16304	.98662	37
24	.09411	.99556	.11147	.99377	.12880	.99167	.14608	.98927	.16333	.98657	36
25	.09440	.99553	.11176	.99374	.12908	.99163	.14637	.98923	.16361	.98652	35
26	.09469	.99551	.11205	.99370	.12937	.99160	.14666	.98919	.16390	.98648	34
27	.09498	.99548	.11234	.99367	.12966	.99156	.14695	.98914	.16419	.98643	33
28	.09527	.99545	.11263	.99364	.12995	.99152	.14723	.98910	.16447	.98638	32
29	.09556	.99542	.11291	.99360	.13024	.99148	.14752	.98906	.16476	.98633	31
30	.09585	.99540	.11320	.99357	.13053	.99144	.14781	.98902	.16505	.98629	30
31	.09614	.99537	.11349	.99354	.13081	.99141	.14810	.98897	.16533	.98624	29
32	.09642	.99534	.11378	.99351	.13110	.99137	.14838	.98893	.16562	.98619	28
33	.09671	.99531	.11407	.99347	.13139	.99133	.14867	.98889	.16591	.98614	27
34	.09700	.99528	.11436	.99344	.13168	.99129	.14896	.98884	.16620	.98609	26
35	.09729	.99526	.11465	.99341	.13197	.99125	.14925	.98880	.16648	.98604	25
36	.09758	.99523	.11494	.99337	.13226	.99122	.14954	.98876	.16677	.98600	24
37	.09787	.99520	.11523	.99334	.13254	.99118	.14982	.98871	.16706	.98595	23
38	.09816	.99517	.11552	.99331	.13283	.99114	.15011	.98867	.16734	.98590	22
39	.09845	.99514	.11580	.99327	.13312	.99110	.15040	.98863	.16763	.98585	21
40	.09874	.99511	.11609	.99324	.13341	.99106	.15069	.98858	.16792	.98580	20
41	.09903	.99508	.11638	.99320	.13370	.99102	.15097	.98854	.16820	.98575	19
42	.09932	.99506	.11667	.99317	.13399	.99098	.15126	.98849	.16849	.98570	18
43	.09961	.99503	.11696	.99314	.13427	.99094	.15155	.98845	.16878	.98565	17
44	.09990	.99500	.11725	.99310	.13456	.99091	.15184	.98841	.16906	.98561	16
45	.10019	.99497	.11754	.99307	.13485	.99087	.15212	.98836	.16935	.98556	15
46	.10048	.99494	.11783	.99303	.13514	.99083	.15241	.98832	.16964	.98551	14
47	.10077	.99491	.11812	.99300	.13543	.99079	.15270	.98827	.16992	.98546	13
48	.10106	.99488	.11840	.99297	.13572	.99075	.15299	.98823	.17021	.98541	12
49	.10135	.99485	.11869	.99293	.13600	.99071	.15327	.98818	.17050	.98536	11
50	.10164	.99482	.11898	.99290	.13629	.99067	.15356	.98814	.17078	.98531	10
51	.10192	.99479	.11927	.99286	.13658	.99063	.15385	.98809	.17107	.98526	9
52	.10221	.99476	.11956	.99283	.13687	.99059	.15414	.98805	.17136	.98521	8
53	.10250	.99473	.11985	.99279	.13716	.99055	.15442	.98800	.17164	.98516	7
54	.10279	.99470	.12014	.99276	.13744	.99051	.15471	.98796	.17193	.98511	6
55	.10308	.99467	.12043	.99272	.13773	.99047	.15500	.98791	.17222	.98506	5
56	.10337	.99464	.12071	.99269	.13802	.99043	.15529	.98787	.17250	.98501	4
57	.10366	.99461	.12100	.99265	.13831	.99039	.15557	.98782	.17279	.98496	3
58	.10395	.99458	.12129	.99262	.13860	.99035	.15586	.98778	.17308	.98491	2
59	.10424	.99455	.12158	.99258	.13889	.99031	.15615	.98773	.17336	.98486	1
60	.10453	.99452	.12187	.99255	.13917	.99027	.15643	.98769	.17365	.98481	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	84°		83°		82°		81°		80°		

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′	10° Sine	10° Cosin	11° Sine	11° Cosin	12° Sine	12° Cosin	13° Sine	13° Cosin	14° Sine	14° Cosin	′
0	.17365	.98481	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	60
1	.17393	.98476	.19109	.98157	.20820	.97809	.22523	.97430	.24220	.97023	59
2	.17422	.98471	.19138	.98152	.20848	.97803	.22552	.97424	.24249	.97015	58
3	.17451	.98466	.19167	.98146	.20877	.97797	.22580	.97417	.24277	.97008	57
4	.17479	.98461	.19195	.98140	.20905	.97791	.22608	.97411	.24305	.97001	56
5	.17508	.98455	.19224	.98135	.20933	.97784	.22637	.97404	.24333	.96994	55
6	.17537	.98450	.19252	.98129	.20962	.97778	.22665	.97398	.24362	.96987	54
7	.17565	.98445	.19281	.98124	.20990	.97772	.22693	.97391	.24390	.96980	53
8	.17594	.98440	.19309	.98118	.21019	.97766	.22722	.97384	.24418	.96973	52
9	.17623	.98435	.19338	.98112	.21047	.97760	.22750	.97378	.24446	.96966	51
10	.17651	.98430	.19366	.98107	.21076	.97754	.22778	.97871	.24474	.96959	50
11	.17680	.98425	.19395	.98101	.21104	.97748	.22807	.97365	.24503	.96952	49
12	.17708	.98420	.19423	.98096	.21132	.97742	.22835	.97358	.24531	.96945	48
13	.17737	.98414	.19452	.98090	.21161	.97735	.22863	.97351	.24559	.96937	47
14	.17766	.98409	.19481	.98084	.21189	.97729	.22892	.97345	.24587	.96930	46
15	.17794	.98404	.19509	.98079	.21218	.97723	.22920	.97338	.24615	.96923	45
16	.17823	.98399	.19538	.98073	.21246	.97717	.22948	.97331	.24644	.96916	44
17	.17852	.98394	.19566	.98067	.21275	.97711	.22977	.97325	.24672	.96909	43
18	.17880	.98389	.19595	.98061	.21303	.97705	.23005	.97318	.24700	.96902	42
19	.17909	.98383	.19623	.98056	.21331	.97698	.23033	.97311	.24728	.96894	41
20	.17937	.98378	.19652	.98050	.21360	.97692	.23062	.97304	.24756	.96887	40
21	.17966	.98373	.19680	.98044	.21388	.97686	.23090	.97298	.24784	.96880	39
22	.17995	.98368	.19709	.98039	.21417	.97680	.23118	.97291	.24813	.96873	38
23	.18023	.98362	.19737	.98033	.21445	.97673	.23146	.97284	.24841	.96866	37
24	.18052	.98357	.19766	.98027	.21474	.97667	.23175	.97278	.24869	.96858	36
25	.18081	.98352	.19794	.98021	.21502	.97661	.23203	.97271	.24897	.96851	35
26	.18109	.98347	.19823	.98016	.21530	.97655	.23231	.97264	.24925	.96844	34
27	.18138	.98341	.19851	.98010	.21559	.97648	.23260	.97257	.24954	.95837	33
28	.18166	.98336	.19880	.98004	.21587	.97642	.23288	.97251	.24982	.96829	32
29	.18195	.98331	.19908	.97997	.21616	.97636	.23316	.97244	.25010	.96822	31
30	.18224	.98325	.19937	.97992	.21644	.97630	.24345	.97237	.25038	.96815	30
31	.18252	.98320	.19965	.97987	.21672	.97623	.23373	.97230	.25066	.96807	29
32	.18281	.98315	.19994	.97981	.21701	.97617	.23401	.97223	.25094	.96800	28
33	.18309	.98310	.20022	.97975	.21729	.97611	.23429	.97217	.25122	.96793	27
34	.18338	.98304	.20051	.97969	.21758	.97604	.23458	.97210	.25151	.96786	26
35	.18367	.98299	.20079	.97963	.21786	.97598	.23486	.97203	.25179	.96778	25
36	.18395	.98294	.20108	.97958	.21814	.97592	.23514	.97196	.25207	.96771	24
37	.18424	.98288	.20136	.97952	.21843	.97585	.23542	.97189	.25235	.96764	23
38	.18452	.98283	.20165	.97946	.21871	.97579	.23571	.97182	.25263	.96756	22
39	.18481	.98277	.20193	.97940	.21899	.97573	.23599	.97176	.25291	.96749	21
40	.18509	.98272	.20222	.97934	.21928	.97566	.23627	.97169	.25320	.96742	20
41	.18538	.98267	.20250	.97928	.21956	.97560	.23656	.97162	.25348	.96734	19
42	.18567	.98261	.20279	.97922	.21985	.97553	.23684	.97155	.25376	.96727	18
43	.18595	.98256	.20307	.97916	.22013	.97547	.23712	.97148	.25404	.96719	17
44	.18624	.98250	.20336	.97910	.22041	.97541	.23740	.97141	.25432	.96712	16
45	.18652	.98245	.20364	.97905	.22070	.97534	.23769	.97134	.25460	.96705	15
46	.18681	.98240	.20393	.97899	.22098	.97528	.23797	.97127	.25488	.96697	14
47	.18710	.98234	.20421	.97893	.22126	.97521	.23825	.97120	.25516	.96690	13
48	.18738	.98229	.20450	.97887	.22155	.97515	.23853	.97113	.25545	.96682	12
49	.18767	.98223	.20478	.97881	.22183	.97508	.23882	.97106	.25573	.96675	11
50	.18795	.98218	.20507	.97875	.22212	.97502	.23910	.97100	.25601	.96667	10
51	.18824	.98212	.20535	.97869	.22240	.97496	.23938	.97093	.25629	.96660	9
52	.18852	.98207	.20563	.97863	.22268	.97489	.23966	.97086	.25657	.96653	8
53	.18881	.98201	.20592	.97857	.22297	.97483	.23995	.97079	.25685	.96645	7
54	.18910	.98196	.20620	.97851	.22325	.97476	.24023	.97072	.25713	.96638	6
55	.18938	.98190	.20649	.97845	.22353	.97470	.24051	.97065	.25741	.96630	5
56	.18967	.98185	.20677	.97839	.22382	.97463	.24079	.97058	.25769	.96623	4
57	.18995	.98179	.20706	.97833	.22410	.97457	.24108	.97051	.25798	.96615	3
58	.19024	.98174	.20734	.97827	.22438	.97450	.24136	.97044	.25826	.96608	2
59	.19052	.98168	.20763	.97821	.22467	.97444	.24164	.97037	.25854	.96600	1
60	.19081	.98163	.20791	.97815	.22495	.97437	.24192	.97030	.25882	.96593	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	79°		78°		77°		76°		75°		

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′	15° Sine	15° Cosine	16° Sine	16° Cosine	17° Sine	17° Cosine	18° Sine	18° Cosine	19° Sine	19° Cosine	′
0	.25882	.96593	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	60
1	.25910	.96585	.27592	.96118	.29265	.95622	.30929	.95097	.32584	.94542	59
2	.25938	.96578	.27620	.96110	.29293	.95613	.30957	.95088	.32612	.94533	58
3	.25966	.96570	.27648	.96102	.29321	.95605	.30985	.95079	.32639	.94523	57
4	.25994	.96562	.27676	.96094	.29348	.95596	.31012	.95070	.32667	.94514	56
5	.26022	.96555	.27704	.96086	.29376	.95588	.31040	.95061	.32694	.94504	55
6	.26050	.96547	.27731	.96078	.29404	.95579	.31068	.95052	.32722	.94495	54
7	.26079	.96540	.27759	.96070	.29432	.95571	.31095	.95043	.32749	.94485	53
8	.26107	.96532	.27787	.96062	.29460	.95562	.31123	.95033	.32777	.94476	52
9	.26135	.96524	.27815	.96054	.29487	.95554	.31151	.95024	.32804	.94466	51
10	.26163	.96517	.27843	.96046	.29515	.95545	.31178	.95015	.32832	.94457	50
11	.26191	.96509	.27871	.96037	.29543	.95536	.31206	.95006	.32859	.94447	49
12	.26219	.96502	.27899	.96029	.29571	.95528	.31233	.94997	.32887	.94438	48
13	.26247	.96494	.27927	.96021	.29599	.95519	.31261	.94988	.32914	.94428	47
14	.26275	.96486	.27955	.96013	.29626	.95511	.31289	.94979	.32942	.94418	46
15	.26303	.96479	.27983	.96005	.29654	.95502	.31316	.94970	.32969	.94409	45
16	.26331	.96471	.28011	.95997	.29682	.95493	.31344	.94961	.32997	.94399	44
17	.26359	.96463	.28039	.95989	.29710	.95485	.31372	.94952	.33024	.94390	43
18	.26387	.96456	.28067	.95981	.29737	.95476	.31399	.94943	.33051	.94380	42
19	.26415	.96448	.28095	.95972	.29765	.95467	.31427	.94933	.33079	.94370	41
20	.26443	.96440	.28123	.95964	.29793	.95459	.31454	.94924	.33106	.94361	40
21	.26471	.96433	.28150	.95956	.29821	.95450	.31482	.94915	.33134	.94351	39
22	.26500	.96425	.28178	.95948	.29849	.95441	.31510	.94906	.33161	.94342	38
23	.26528	.96417	.28206	.95940	.29876	.95433	.31537	.94897	.33189	.94332	37
24	.26556	.96410	.28234	.95931	.29904	.95424	.31565	.94888	.33216	.94322	36
25	.26584	.96402	.28262	.95923	.29932	.95415	.31593	.94878	.33244	.94313	35
26	.26612	.96394	.28290	.95915	.29960	.95407	.31620	.94869	.33271	.94303	34
27	.26640	.96386	.28318	.95907	.29987	.95398	.31648	.94860	.33298	.94293	33
28	.26668	.96379	.28346	.95898	.30015	.95389	.31675	.94851	.33326	.94284	32
29	.26696	.96371	.28374	.95890	.30043	.95380	.31703	.94842	.33353	.94274	31
30	.26724	.96363	.28402	.95882	.30071	.95372	.31730	.94832	.33381	.94264	30
31	.26752	.96355	.28429	.95874	.30098	.95363	.31758	.94823	.33408	.94254	29
32	.26780	.90347	.28457	.95865	.30126	.95354	.31786	.94814	.33436	.94245	28
33	.26808	.96340	.28485	.95857	.30154	.95345	.31813	.94805	.33463	.94235	27
34	.26836	.96332	.28513	.95849	.30182	.95337	.31841	.94795	.33490	.94225	26
35	.26864	.96324	.28541	.95841	.30209	.95328	.31868	.94786	.33518	.94215	25
36	.26892	.96316	.28569	.95832	.30237	.95319	.31896	.94777	.33545	.94206	24
37	.26920	.96308	.28597	.95824	.30265	.95310	.31923	.94768	.33573	.94196	23
38	.26948	.96301	.28625	.95816	.30292	.95301	.31951	.94758	.33600	.94186	22
39	.26976	.96293	.28652	.95807	.30320	.95293	.31979	.94749	.33627	.94176	21
40	.27004	.96285	.28680	.95799	.30348	.95284	.32006	.94740	.33655	.94167	20
41	.27032	.96277	.28708	.95791	.30376	.95275	.32034	.94730	.33682	.94157	19
42	.27060	.96269	.28736	.95782	.30403	.95266	.32061	.94721	.33710	.94147	18
43	.27088	.96261	.28764	.95774	.30431	.95257	.32089	.94712	.33737	.94137	17
44	.27116	.96253	.28792	.95766	.30459	.95248	.32116	.94702	.33764	.94127	16
45	.27144	.96246	.28820	.95757	.30486	.95240	.32144	.94693	.33792	.94118	15
46	.27172	.96238	.28847	.95749	.30514	.94231	.32171	.94684	.33819	.94108	14
47	.27200	.96230	.28875	.95740	.30542	.95222	.32199	.94674	.33846	.94098	13
48	.27228	.96222	.28903	.95732	.30570	.95213	.32227	.94665	.33874	.94088	12
49	.27256	.96214	.28931	.95724	.30597	.95204	.32254	.94656	.33901	.94078	11
50	.27284	.96206	.28959	.95715	.30625	.95195	.32282	.94646	.33929	.94068	10
51	.27312	.96198	.28987	.95707	.30653	.95186	.32309	.94637	.33956	.94058	9
52	.27340	.96190	.29015	.95698	.30680	.95177	.32337	.94627	.33983	.94049	8
53	.27368	.96182	.29042	.95690	.30708	.95168	.32364	.94618	.34011	.94039	7
54	.27396	.96174	.29070	.95681	.30736	.95159	.32392	.94609	.34038	.94029	6
55	.27424	.96166	.29098	.95673	.30763	.95150	.32419	.94599	.34065	.94019	5
56	.27452	.96158	.29126	.95664	.30791	.95142	.32447	.94590	.34093	.94009	4
57	.27480	.96150	.29154	.95656	.30819	.95133	.32474	.94580	.34120	.93999	3
58	.27508	.96142	.29182	.95647	.30846	.95124	.32502	.94571	.34147	.93989	2
59	.27536	.96134	.29209	.95639	.30874	.95115	.32529	.94561	.34175	.93979	1
60	.27564	.96126	.29237	.95630	.30902	.95106	.32557	.94552	.34202	.93969	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	74°		73°		72°		71°		70°		

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′	20° Sine	20° Cosine	21° Sine	21° Cosine	22° Sine	22° Cosine	23° Sine	23° Cosine	24° Sine	24° Cosine	′
0	.34202	.93969	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	60
1	.34229	.93959	.35864	.93348	.37488	.92707	.39100	.92039	.40700	.91343	59
2	.34257	.93949	.35891	.93337	.37515	.92697	.39127	.92028	.40727	.91331	58
3	.34284	.93939	.35918	.93327	.37542	.92686	.39153	.92016	.40753	.91319	57
4	.34311	.93929	.35945	.93316	.37569	.92675	.39180	.92005	.40780	.91307	56
5	.34339	.93919	.35973	.93306	.37595	.92664	.39207	.91994	.40806	.91295	55
6	.34366	.93909	.36000	.93295	.37622	.92653	.39234	.91982	.40833	.91283	54
7	.34393	.93899	.36027	.93285	.37649	.92642	.39260	.91971	.40860	.91272	53
8	.34421	.93889	.36054	.93274	.37676	.92631	.39287	.91959	.40886	.91260	52
9	.34448	.93879	.36081	.93264	.37703	.92620	.39314	.91948	.40913	.91248	51
10	.34475	.93869	.36108	.93253	.37730	.92609	.39341	.91936	.40939	.91236	50
11	.34503	.93859	.36135	.93243	.37757	.92598	.39367	.91925	.40966	.91224	49
12	.34530	.93849	.36162	.93232	.37784	.92587	.39394	.91914	.40992	.91212	48
13	.34557	.93839	.36190	.93222	.37811	.92576	.39421	.91902	.41019	.91200	47
14	.34584	.93829	.36217	.93211	.37838	.92565	.39448	.91891	.41045	.91188	46
15	.34612	.93819	.36244	.93201	.37865	.92554	.39474	.91879	.41072	.91176	45
16	.34639	.93809	.36271	.93190	.37892	.92543	.39501	.91868	.41098	.91164	44
17	.34666	.93799	.36298	.93180	.37919	.92532	.39528	.91856	.41125	.91152	43
18	.34694	.93789	.36325	.93169	.37946	.92521	.39555	.91845	.41151	.91140	42
19	.34721	.93779	.36352	.93159	.37973	.92510	.39581	.91833	.41178	.91128	41
20	.34748	.93769	.36379	.93148	.37999	.92499	.39608	.91822	.41204	.91116	40
21	.34775	.93759	.36406	.93137	.38026	.92488	.39635	.91810	.41231	.91104	39
22	.34803	.93748	.36434	.93127	.38053	.92477	.39661	.91799	.41257	.91092	38
23	.34830	.93738	.36461	.93116	.38080	.92466	.39688	.91787	.41284	.91080	37
24	.34857	.93728	.36488	.93106	.38107	.92455	.39715	.91775	.41310	.91068	36
25	.34884	.93718	.36515	.93095	.38134	.92544	.39741	.91764	.41337	.91056	35
26	.34912	.93708	.36542	.93084	.38161	.92432	.39768	.91752	.41363	.91044	34
27	.34939	.93698	.36569	.93074	.38188	.92421	.39795	.91741	.41390	.91032	33
28	.34966	.93688	.36596	.93063	.38215	.02410	.39822	.91729	.41416	.91020	32
29	.34993	.93677	.36623	.93052	.38241	.92399	.39848	.91718	.41443	.91008	31
30	.35021	.93667	.36650	.93042	.38268	.92388	.39875	.91706	.41469	.90996	30
31	.35048	.93657	.36677	.93031	.38295	.92377	.39902	.91694	.41496	.90984	29
32	.35075	.93647	.36704	.93020	.38322	.92366	.39928	.91683	.41522	.90972	28
33	.35102	.93637	.36731	.93010	.38349	.92355	.39955	.91671	.41549	.90960	27
34	.35130	.93626	.36758	.92999	.38376	.92343	.39982	.91660	.41575	.90948	26
35	.35157	.93616	.36785	.92988	.38403	.92332	.40008	.91648	.41602	.90936	25
36	.35184	.93606	.36812	.92978	.38430	.92321	.40035	.91636	.41628	.90924	24
37	.35211	.93596	.36839	.92967	.38456	.92310	.40062	.91625	.41655	.90911	23
38	.35239	.93585	.36867	.92956	.38483	.92299	.40088	.91613	.41681	.90899	22
39	.35266	.93575	.36894	.92945	.28510	.92287	.40115	.91601	.41707	.90887	21
40	.35293	.93565	.36921	.92935	.38537	.92276	.40141	.91590	.41734	.90875	20
41	.35320	.93555	.36948	.92924	.38564	.92265	.40168	.91578	.41760	.90863	19
42	.35347	.93544	.36975	.92913	.38591	.92254	.40195	.91566	.41787	.90851	18
43	.35375	.93534	.37002	.92902	.38617	.92243	.40221	.91555	.41813	.90839	17
44	.35402	.93524	.37029	.92892	.38644	.92231	.40248	.91543	.41840	.90826	16
45	.35429	.93514	.37056	.92881	.38671	.92220	.40275	.91531	.41866	.90814	15
46	.35456	.93503	.37083	.92870	.38698	.92209	.40301	.91519	.41892	.90802	14
47	.35484	.93493	.37110	.92859	.38725	.92198	.40328	.91508	.41919	.90790	13
48	.35511	.93483	.37137	.92849	.38752	.92186	.40355	.91496	.41945	.90778	12
49	.35538	.93472	.37164	.92838	.38778	.92175	.40381	.91484	.41972	.90766	11
50	.35565	.93462	.37191	.92827	.38805	.92164	.40408	.91472	.41998	.90753	10
51	.35592	.93452	.37218	.92816	.38832	.92152	.40434	.91461	.42024	.90741	9
52	.35619	.93441	.37245	.92805	.38859	.92141	.40461	.91449	.42051	.90729	8
53	.35647	.93431	.37272	.92794	.38886	.92130	.40488	.91437	.42077	.90717	7
54	.35674	.93420	.37299	.92784	.38912	.92119	.40514	.91425	.42104	.90704	6
55	.35701	.93410	.37326	.92773	.38939	.92107	.40541	.91414	.42130	.90692	5
56	.35728	.93400	.37353	.92762	.38966	.92096	.40567	.91402	.42156	.90680	4
57	.35755	.93389	.37380	.92751	.38993	.92085	.40594	.91390	.42183	.90668	3
58	.35782	.93379	.37407	.92740	.39020	.92073	.40621	.91378	.42209	.90655	2
59	.35810	.93368	.37434	.92729	.39046	.92062	.40647	.91366	.42235	.90643	1
60	.35837	.93358	.37461	.92718	.39073	.92050	.40674	.91355	.42262	.90631	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	69°		68°		67°		66°		65°		

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′	25° Sine	25° Cosin	26° Sine	26° Cosin	27° Sine	27° Cosin	28° Sine	28° Cosin	29° Sine	29° Cosin	′
0	.42262	.90631	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	60
1	.42288	.90618	.43863	.89867	.45425	.89087	.46973	.88281	.48506	.87448	59
2	.42315	.90606	.43889	.89854	.45451	.89074	.46999	.88267	.48532	.87434	58
3	.42341	.90594	.43916	.89841	.45477	.89061	.47024	.88254	.48557	.87420	57
4	.42367	.90582	.43942	.89828	.45503	.89048	.47050	.88240	.48583	.87406	56
5	.42394	.90569	.43968	.89816	.45529	.89035	.47076	.88226	.48608	.87391	55
6	.42420	.90557	.43994	.89803	.45554	.89021	.47101	.88213	.48634	.87377	54
7	.42446	.90545	.44020	.89790	.45580	.89008	.47127	.88199	.48659	.87363	53
8	.42473	.90532	.44046	.89777	.45606	.88995	.47153	.88185	.48684	.87349	52
9	.42499	.90520	.44072	.89764	.45632	.88981	.47178	.88172	.48710	.87335	51
10	.42525	.90507	.44098	.89752	.45658	.88968	.47204	.88158	.48735	.87321	50
11	.42552	.90495	.44124	.80739	.45684	.88955	.47229	.88144	.48761	.87306	49
12	.42578	.90483	.44151	.89726	.45710	.88942	.47255	.88130	.48786	.87292	48
13	.42604	.90470	.44177	.89713	.45736	.88928	.47281	.88117	.48811	.87278	47
14	.42631	.90458	.44203	.89700	.45762	.88915	.47306	.88103	.48837	.87264	46
15	.42657	.90446	.44229	.89687	.45787	.88902	.47332	.88089	.48862	.87250	45
16	.42683	.90433	.44255	.89674	.45813	.88888	.47358	.88075	.48888	.87235	44
17	.42709	.90421	.44281	.89662	.45839	.88875	.47383	.88062	.48913	.87221	43
18	.42736	.90408	.44307	.89649	.45865	.88862	.47409	.88048	.48938	.87207	42
19	.42762	.90396	.44333	.89636	.45891	.88848	.47434	.88034	.48964	.87193	41
20	.42788	.90383	.44359	.89623	.45917	.88835	.47460	.88020	.48989	.87178	40
21	.42815	.90371	.44385	.89610	.45942	.88822	.47486	.88006	.49014	.87164	39
22	.42841	.90358	.44411	.89597	.45968	.88808	.47511	.87993	.49040	.87150	38
23	.42867	.90346	.44437	.89584	.45994	.88795	.47537	.87979	.49065	.87136	37
24	.42894	.90334	.44464	.89571	.46020	.88782	.47562	.87965	.49090	.87121	36
25	.42920	.90321	.44490	.89558	.46046	.88768	.47588	.87951	.49116	.87107	35
26	.42946	.90309	.44516	.89545	.46072	.88755	.47614	.87937	.47141	.87093	34
27	.42972	.90296	.44542	.89532	.46097	.88741	.47639	.87923	.49166	.87079	33
28	.42999	.90284	.44568	.89519	.46123	.88728	.47665	.87909	.49192	.87064	32
29	.43025	.90271	.44594	.89506	.46149	.88715	.47690	.87896	.49217	.87050	31
30	.43051	.90259	.44620	.89493	.46175	.88701	.47716	.87882	.49242	.87036	30
31	.43077	.90246	.44646	.89480	.46201	.88688	.47741	.87868	.49268	.87021	29
32	.43104	.90233	.44672	.89467	.46226	.88674	.47767	.87854	.49293	.87007	28
33	.43130	.90221	.44698	.89454	.46252	.88661	.47793	.87840	.49318	.86993	27
34	.43156	.90208	.44724	.89441	.46278	.88647	.47818	.87826	.49344	.86978	26
35	.43182	.90196	.44750	.89428	.46304	.88634	.47844	.87812	.49369	.86964	25
36	.43209	.90183	.44776	.89415	.46330	.88620	.47869	.87798	.49394	.86949	24
37	.43235	.90171	.44802	.89402	.46355	.88607	.47895	.87784	.49419	.86935	23
38	.43261	.90158	.44828	.89389	.46381	.88593	.47920	.87770	.49445	.86921	22
39	.43287	.90146	.44854	.89376	.46407	.88580	.47946	.87756	.49470	.86906	21
40	.43313	.90133	.44880	.89363	.46433	.88566	.47971	.87743	.49495	.86892	20
41	.43340	.90120	.44906	.89350	.46458	.88553	.47997	.87729	.49521	.86878	19
42	.43366	.90108	.44932	.89337	.46484	.88539	.48022	.87715	.49546	.86863	18
43	.43392	.90095	.44958	.89324	.46510	.88526	.48048	.87701	.49571	.86849	17
44	.43418	.90082	.44984	.89311	.46536	.88512	.48073	.87687	.49596	.86834	16
45	.43445	.90070	.45010	.89298	.46561	.88499	.48099	.87673	.49622	.86820	15
46	.43471	.90057	.45036	.89285	.46587	.88485	.48124	.87659	.49647	.86805	14
47	.43497	.90045	.45062	.89272	.46613	.88472	.48150	.87645	.49672	.86791	13
48	.43523	.90032	.45088	.89259	.46639	.88458	.48175	.87631	.49697	.86777	12
49	.43549	.90019	.45114	.89245	.46664	.88445	.48201	.87617	.49723	.86762	11
50	.43575	.90007	.45140	.89232	.46690	.88431	.48226	.87603	.49748	.86748	10
51	.43602	.89994	.45166	.89219	.46716	.88417	.48252	.87589	.49773	.86733	9
52	.43628	.89981	.45192	.89206	.46742	.88404	.48277	.87575	.49798	.86719	8
53	.43654	.89968	.45218	.89193	.46767	.88390	.48303	.87561	.49824	.86704	7
54	.43680	.89956	.45243	.89180	.46793	.88377	.48328	.87546	.49849	.86690	6
55	.43706	.89943	.45269	.89167	.46819	.88363	.48354	.87532	.49874	.86675	5
56	.43733	.89930	.45295	.89153	.46844	.88349	.48379	.87518	.49899	.86661	4
57	.43759	.89918	.45321	.89140	.46870	.88336	.48405	.87504	.49924	.86646	3
58	.43785	.89905	.45347	.89127	.46896	.88322	.48430	.87490	.49950	.86632	2
59	.43811	.89892	.45373	.89114	.46921	.88308	.48456	.87476	.49975	.86617	1
60	.43837	.89879	.45399	.89101	.46947	.88295	.48481	.87462	.50000	.86603	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	64°		63°		62°		61°		60°		

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′	30° Sine	30° Cosin	31° Sine	31° Cosin	32° Sine	32° Cosin	33° Sine	33° Cosin	34° Sine	34° Cosin	′
0	.50000	.86603	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	60
1	.50025	.86588	.51529	.85702	.53017	.84789	.54488	.83851	.55943	.82887	59
2	.50050	.86573	.51554	.85687	.53041	.84774	.54513	.83835	.55968	.82871	58
3	.50076	.86559	.51579	.85672	.53066	.84759	.54537	.83819	.55992	.82855	57
4	.50101	.86544	.51604	.85657	.53091	.84743	.54561	.83804	.56016	.82839	56
5	.50126	.86530	.51628	.85642	.53115	.84728	.54586	.83788	.56040	.82822	55
6	.50151	.86515	.51653	.85627	.53140	.84712	.54610	.83772	.56064	.82806	54
7	.50176	.86501	.51678	.85612	.53164	.84697	.54635	.83756	.56088	.82790	53
8	.50201	.86486	.51703	.85597	.53189	.84681	.54659	.83740	.56112	.82773	52
9	.50227	.86471	.51728	.85582	.53214	.84666	.54683	.83724	.56136	.82757	51
10	.50252	.86457	.51753	.85567	.53238	.84650	.54708	.83708	.56160	.82741	50
11	.50277	.86442	.51778	.85551	.53263	.84635	.54732	.83692	.56184	.82724	49
12	.50302	.86427	.51803	.85536	.53288	.84619	.54756	.83676	.56208	.82708	48
13	.50327	.86413	.51828	.85521	.53312	.84604	.54781	.83660	.56232	.82692	47
14	.50352	.86398	.51852	.85506	.53337	.84588	.54805	.83645	.56256	.82675	46
15	.50377	.86384	.51877	.85491	.53361	.84573	.54829	.83629	.56280	.82659	45
16	.50403	.86369	.51902	.85476	.53386	.84557	.54854	.83613	.56305	.82643	44
17	.50428	.86354	.51927	.85461	.53411	.84542	.54878	.83597	.56329	.82626	43
18	.50453	.86340	.51952	.85446	.53435	.84526	.54902	.83581	.56353	.82610	42
19	.50478	.86325	.51977	.85431	.53460	.84511	.54927	.83565	.56377	.82593	41
20	.50503	.86310	.52002	.85416	.53484	.84495	.54951	.83549	.56401	.82577	40
21	.50528	.86295	.52026	.85401	.53509	.84480	.54975	.83533	.56425	.82561	39
22	.50553	.86281	.52051	.85385	.53534	.84464	.54999	.83517	.56449	.82544	38
23	.50578	.86266	.52076	.85370	.53558	.84448	.55024	.83501	.56473	.82528	37
24	.50603	.86251	.52101	.85355	.53583	.84433	.55048	.83485	.56497	.82511	36
25	.50628	.86237	.52126	.85340	.53607	.84417	.55072	.83469	.56521	.82495	35
26	.50654	.86222	.52151	.85325	.53632	.84402	.55097	.83453	.56545	.82478	34
27	.50679	.86207	.52175	.85310	.53656	.84386	.55121	.83437	.56569	.82462	33
28	.50704	.86192	.52200	.85294	.53681	.84370	.55145	.83421	.56593	.82446	32
29	.50729	.86178	.52225	.85279	.53705	.84355	.55169	.83405	.56617	.82429	31
30	.50754	.86163	.52250	.85264	.53730	.84339	.55194	.83389	.56641	.82413	30
31	.50779	.86148	.52275	.85249	.53754	.84324	.55218	.83373	.56665	.82396	29
32	.50804	.86133	.52299	.85234	.53779	.84308	.55242	.83356	.56689	.82380	28
33	.50829	.86119	.52324	.85218	.53804	.84292	.55266	.83340	.56713	.82363	27
34	.50854	.86104	.52349	.85203	.53828	.84277	.55291	.83324	.56736	.82347	26
35	.50879	.86089	.52374	.85188	.53853	.84261	.55315	.83308	.56760	.82330	25
36	.50904	.86074	.52399	.85173	.53877	.84245	.55339	.83292	.56784	.82314	24
37	.50929	.86059	.52423	.85157	.53902	.84230	.55363	.83276	.56808	.82297	23
38	.50954	.86045	.52448	.85142	.53926	.84214	.55388	.83260	.56832	.82281	22
39	.50979	.86030	.52473	.85127	.53951	.84198	.55412	.83244	.56856	.82264	21
40	.51004	.86015	.52498	.85112	.53975	.84182	.55436	.83228	.56880	.82248	20
41	.51029	.86000	.52522	.85096	.54000	.84167	.55460	.83212	.56904	.82231	19
42	.51054	.85985	.52547	.85081	.54024	.84151	.55484	.83195	.56928	.82214	18
43	.51079	.85970	.52572	.85066	.54049	.84135	.55509	.83179	.56952	.82198	17
44	.51104	.85956	.52597	.85051	.54073	.84120	.55533	.83163	.56976	.82181	16
45	.51129	.85941	.52621	.85035	.54097	.84104	.55557	.83147	.57000	.82165	15
46	.51154	.85926	.52646	.85020	.54122	.84088	.55581	.83131	.57024	.82148	14
47	.51179	.85911	.52671	.85005	.54146	.84072	.55605	.83115	.57047	.82132	13
48	.51204	.85896	.52696	.84989	.54171	.84057	.55630	.83098	.57071	.82115	12
49	.51229	.85881	.52720	.84974	.54195	.84041	.55654	.83082	.57095	.82098	11
50	.51254	.85866	.52745	.84959	.54220	.84025	.55678	.83066	.57119	.82082	10
51	.51279	.85851	.52770	.84943	.54244	.84009	.55702	.83050	.57143	.82065	9
52	.51304	.85836	.52794	.84928	.54269	.83994	.55726	.83034	.57167	.82048	8
53	.51329	.85821	.52819	.84913	.54293	.83978	.55750	.83017	.57191	.82032	7
54	.51354	.85806	.52844	.84897	.54317	.83962	.55775	.83001	.57215	.82015	6
55	.51379	.85792	.52869	.84882	.54342	.83946	.55799	.82985	.57238	.81999	5
56	.51404	.85777	.52893	.84866	.54366	.83930	.55823	.82969	.57262	.81982	4
57	.51429	.85762	.52918	.84851	.54391	.83915	.55847	.82953	.57286	.81965	3
58	.51454	.85747	.52943	.84836	.54415	.83899	.55871	.82936	.57310	.81949	2
59	.51479	.85732	.52967	.84820	.54440	.83883	.55895	.82920	.57334	.81932	1
60	.51504	.85717	.52992	.84805	.54464	.83867	.55919	.82904	.57358	.81915	0
′	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	Cosin	Sine	′
	59°		58°		57°		56°		55°		

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′	35° Sine	35° Cosin	36° Sine	36° Cosin	37° Sine	37° Cosin	38° Sine	38° Cosin	39° Sine	39° Cosin	′
0	.57358	.81915	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	60
1	.57381	.81899	.58802	.80885	.60205	.79846	.61589	.78783	.62955	.77696	59
2	.57405	.81882	.58826	.80867	.60228	.79829	.61612	.78765	.62977	.77678	58
3	.57429	.81865	.58849	.80850	.60251	.79811	.61635	.78747	.63000	.77660	57
4	.57453	.81848	.58873	.80833	.60274	.79793	.61658	.78729	.63022	.77641	56
5	.57477	.81832	.58896	.80816	.60298	.79776	.61681	.78711	.63045	.77623	55
6	.57501	.81815	.58920	.80799	.60321	.79758	.61704	.78694	.63068	.77605	54
7	.57524	.81798	.58943	.80782	.60344	.79741	.61726	.78676	.63090	.77586	53
8	.57548	.81782	.58967	.80765	.60367	.79723	.61749	.78658	.63113	.77568	52
9	.57572	.81765	.58990	.80748	.60390	.79706	.61772	.78640	.63135	.77550	51
10	.57596	.81748	.59014	.80730	.60414	.79688	.61795	.78622	.68158	.77531	50
11	.57619	.81731	.59037	.80713	.60437	.79671	.61818	.78604	.63180	.77513	49
12	.57643	.81714	.59061	.80696	.60460	.79653	.61841	.78586	.63203	.77494	48
13	.57667	.81698	.59084	.80679	.60483	.79635	.61864	.78568	.63225	.77476	47
14	.57691	.81681	.59108	.80662	.60506	.79618	.61887	.78550	.63248	.77458	46
15	.57715	.81664	.59131	.80644	.60529	.79600	.61909	.78532	.63271	.77439	45
16	.57738	.81647	.59154	.80627	.60553	.79583	.61932	.78514	.63293	.77421	44
17	.57762	.81631	.59178	.80610	.60576	.79565	.61955	.78496	.63316	.77402	43
18	.57786	.81614	.59201	.80593	.60599	.79547	.61978	.78478	.63338	.77384	42
19	.57810	.81597	.59225	.80576	.60622	.79530	.62001	.78460	.63361	.77366	41
20	.57833	.81580	.59248	.80558	.60645	.79512	.62024	.78442	.63383	.77347	40
21	.57857	.81563	.59272	.80541	.60668	.79494	.62046	.78424	.63406	.77329	39
22	.57881	.81546	.59295	.80524	.60691	.79477	.62069	.78405	.63428	.77310	38
23	.57904	.81530	.59318	.80507	.60714	.79459	.62092	.78387	.63451	.77292	37
24	.57928	.81513	.59342	.80489	.60738	.79441	.62115	.78369	.63473	.77273	36
25	.57952	.81496	.59365	.80472	.60761	.79424	.62138	.78351	.63496	.77255	35
26	.57976	.81479	.59389	.80455	.60784	.79406	.62160	.78333	.63518	.77236	34
27	.57999	.81462	.59412	.80438	.60807	.79388	.62183	.78315	.63540	.77218	33
28	.58023	.81445	.59436	.80420	.60830	.79371	.62206	.78297	.63563	.77199	32
29	.58047	.81428	.59459	.80403	.60853	.79353	.62229	.78279	.63585	.77181	31
30	.58070	.81412	.59482	.80386	.60876	.79335	.62251	.78261	.63608	.77162	30
31	.58094	.81395	.59506	.80368	.60899	.79318	.62274	.78243	.63630	.77144	29
32	.58118	.81378	.59529	.80351	.60922	.79300	.62297	.78225	.63653	.77125	28
33	.58141	.81361	.59552	.80334	.60945	.79282	.62320	.78206	.63675	.77107	27
34	.58165	.81344	.59576	.80316	.60968	.79264	.62342	.78188	.63698	.77088	26
35	.58189	.81327	.59599	.80299	.60991	.79247	.62365	.78170	.63720	.77070	25
36	.58212	.81310	.59622	.80282	.61015	.79229	.62388	.78152	.63742	.77051	24
37	.58236	.81293	.59646	.80264	.61038	.79211	.62411	.78134	.63765	.77033	23
38	.58260	.81276	.59669	.80247	.61061	.79193	.62433	.78116	.63787	.77014	22
39	.58283	.81259	.59693	.80230	.61084	.79176	.62456	.78098	.63810	.76996	21
40	.58307	.81242	.59716	.80212	.61107	.79158	.62479	.78079	.63832	.76977	20
41	.58330	.81225	.59739	.80195	.61130	.79140	.62502	.78061	.63854	.76959	19
42	.58354	.81208	.59763	.80178	.61153	.79122	.62524	.78043	.63877	.76940	18
43	.58378	.81191	.59786	.80160	.61176	.79105	.62547	.78025	.63899	.76921	17
44	.58401	.81174	.59809	.80143	.61199	.79087	.62570	.78007	.63922	.76903	16
45	.58425	.81157	.59832	.80125	.61222	.79069	.62592	.77988	.63944	.76884	15
46	.58449	.81140	.59856	.80108	.61245	.79051	.62615	.77970	.63966	.76866	14
47	.58472	.81123	.59879	.80091	.61268	.79033	.62638	.77952	.63989	.76847	13
48	.58496	.81106	.59902	.80073	.61291	.79016	.62660	.77934	.64011	.76828	12
49	.58519	.81089	.59926	.80056	.61314	.78998	.62683	.77916	.64033	.76810	11
50	.58543	.81072	.59949	.80038	.61337	.78980	.62706	.77897	.64056	.76791	10
51	.58567	.81055	.59972	.80021	.61360	.78962	.62728	.77879	.64078	.76772	9
52	.58590	.81038	.59995	.80003	.61383	.78944	.62751	.77861	.64100	.76754	8
53	.58614	.81021	.60019	.79986	.61406	.78926	.62774	.77843	.64123	.76735	7
54	.58637	.81004	.60042	.79968	.61429	.78908	.62796	.77824	.64145	.76717	6
55	.58661	.80987	.60065	.79951	.61451	.78891	.62819	.77806	.64167	.76698	5
56	.58684	.80970	.60089	.79934	.61474	.78873	.62842	.77788	.64190	.76679	4
57	.58708	.80953	.60112	.79916	.61497	.78855	.62864	.77769	.64212	.76661	3
58	.58731	.80936	.60135	.79899	.61520	.78837	.62887	.77751	.64234	.76642	2
59	.58755	.80919	.60158	.79881	.61543	.78819	.62909	.77733	.64256	.76623	1
60	.58779	.80902	.60182	.79864	.61566	.78801	.62932	.77715	.64279	.76604	0
′	54° Cosin	54° Sine	53° Cosin	53° Sine	52° Cosin	52° Sine	51° Cosin	51° Sine	50° Cosin	50° Sine	′

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′	40° Sine	40° Cosin	41° Sine	41° Cosin	42° Sine	42° Cosin	43° Sine	43° Cosin	44° Sine	44° Cosin	′
0	.64279	.76604	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	60
1	.64301	.76586	.65628	.75452	.66935	.74295	.68221	.73116	.69487	.71914	59
2	.64323	.76567	.65650	.75433	.66956	.74276	.68242	.73096	.69508	.71894	58
3	.64346	.76548	.65672	.75414	.66978	.74256	.68264	.73076	.69529	.71873	57
4	.64368	.76530	.65694	.75395	.66999	.74237	.68285	.73056	.69549	.71853	56
5	.64390	.76511	.65716	.75375	.67021	.74217	.68306	.73036	.69570	.71833	55
6	.64412	.76492	.65738	.75356	.67043	.74198	.68327	.73016	.69591	.71813	54
7	.64435	.76473	.65759	.75337	.67064	.74178	.68349	.72996	.69612	.71792	53
8	.64457	.76455	.65781	.75318	.67086	.74159	.68370	.72976	.69633	.71772	52
9	.64479	.76436	.65803	.75299	.67107	.74139	.68391	.72957	.69654	.71752	51
10	.64501	.76417	.65825	.75280	.67129	.74120	.68412	.72937	.69675	.71732	50
11	.64524	.76398	.65847	.75261	.67151	.74100	.68434	.72917	.69696	.71711	49
12	.64546	.76380	.65869	.75241	.67172	.74080	.68455	.72897	.69717	.71691	48
13	.64568	.76361	.65891	.75222	.67194	.74061	.68476	.72877	.69737	.71671	47
14	.64590	.76342	.65913	.75203	.67215	.74041	.68497	.72857	.69758	.71650	46
15	.64612	.76323	.65935	.75184	.67237	.74022	.68518	.72837	.69779	.71630	45
16	.64635	.76304	.65956	.75165	.67258	.74002	.68539	.72817	.69800	.71610	44
17	.64657	.76286	.65978	.75146	.67280	.73983	.68561	.72797	.69821	.71590	43
18	.64679	.76267	.66000	.75126	.67301	.73963	.68582	.72777	.69842	.71569	42
19	.64701	.76248	.66022	.75107	.67323	.73944	.68603	.72757	.69862	.71549	41
20	.64723	.76229	.66044	.75088	.67344	.73924	.68624	.72737	.69883	.71529	40
21	.64746	.76210	.66066	.75069	.67366	.73904	.68645	.72717	.69904	.71508	39
22	.64768	.76192	.66088	.75050	.67387	.73885	.68666	.72697	.69925	.71488	38
23	.64790	.76173	.66109	.75030	.67409	.73865	.68688	.72677	.69946	.71468	37
24	.64812	.76154	.66131	.75011	.67430	.73846	.68709	.72657	.69966	.71447	36
25	.64834	.76135	.66153	.74992	.67452	.73826	.68730	.72637	.69987	.71427	35
26	.64856	.76116	.66175	.74973	.67473	.73806	.68751	.72617	.70008	.71407	34
27	.64878	.76097	.66197	.74953	.67495	.73787	.69772	.72597	.70029	.71386	33
28	.64901	.76078	.66218	.74934	.67516	.73767	.68793	.72577	.70049	.71366	32
29	.64923	.76059	.66240	.74915	.67538	.73747	.68814	.72557	.70070	.71345	31
30	.64945	.76041	.66262	.74896	.67559	.73728	.68835	.72537	.70091	.71325	30
31	.64967	.76022	.66284	.74876	.67580	.73708	.68857	.72517	.70112	.71305	29
32	.64989	.76003	.66306	.74857	.67602	.73683	.68878	.72497	.70132	.71284	28
33	.65011	.75984	.66327	.74838	.67623	.73669	.68899	.72477	.70153	.71264	27
34	.65033	.75965	.66349	.74818	.67645	.73649	.68920	.72457	.70174	.71243	26
35	.65055	.75946	.66371	.74799	.67666	.73629	.68941	.72437	.70195	.71223	25
36	.65077	.75927	.66393	.74780	.67688	.73610	.68962	.72417	.70215	.71203	24
37	.65100	.75908	.66414	.74760	.67709	.73590	.68983	.72397	.70236	.71182	23
38	.65122	.75889	.66436	.74741	.67730	.73570	.69004	.73377	.70257	.71162	22
39	.65144	.75870	.66458	.74722	.67752	.73551	.69025	.72357	.70277	.71141	21
40	.65166	.75851	.66480	.74703	.67773	.73531	.69046	.72337	.70298	.71121	20
41	.65188	.75832	.66501	.74683	.67795	.73511	.69057	.72317	.70319	.71100	19
42	.65210	.75813	.66523	.74664	.67816	.73491	.69088	.72297	.70339	.71080	18
43	.65232	.75794	.66545	.74644	.67837	.73472	.69109	.72277	.70360	.71059	17
44	.65254	.75775	.66566	.74625	.67859	.73452	.69130	.72257	.70381	.71039	16
45	.65276	.75756	.66588	.74606	.67880	.73432	.69151	.72236	.70401	.71019	15
46	.65298	.75738	.66610	.74586	.67901	.73413	.69172	.72216	.70422	.70998	14
47	.65320	.75719	.66632	.74567	.67923	.73393	.69193	.72196	.70443	.70978	13
48	.65342	.75700	.66653	.74548	.67944	.73373	.69214	.72176	.70463	.70957	12
49	.65364	.75680	.66675	.74528	.67965	.73353	.69235	.72156	.70484	.70937	11
50	.65386	.75661	.66697	.74509	.67987	.73333	.69256	.72136	.70505	.70916	10
51	.65408	.75642	.66718	.74489	.68008	.73314	.69277	.72116	.70525	.70896	9
52	.65430	.75623	.66740	.74470	.68029	.73294	.69298	.72095	.70546	.70875	8
53	.65452	.75604	.66762	.74451	.68051	.73274	.69319	.72075	.70567	.70855	7
54	.65474	.75585	.66783	.74431	.68072	.73254	.69340	.72055	.70587	.70834	6
55	.65496	.75566	.66805	.74412	.68093	.73234	.69361	.72035	.70608	.70813	5
56	.65518	.75547	.66827	.74392	.68115	.73215	.69382	.72015	.70628	.70793	4
57	.65540	.75528	.66848	.74373	.68136	.73195	.69403	.71995	.70649	.70772	3
58	.65562	.75509	.66870	.74353	.68157	.73175	.69424	.71974	.70670	.70752	2
59	.65584	.75490	.66891	.74334	.68179	.73155	.69445	.71954	.70690	.70731	1
60	.65606	.75471	.66913	.74314	.68200	.73135	.69466	.71934	.70711	.70711	0
′	49° Cosin	49° Sine	48° Cosin	48° Sine	47° Cosin	47° Sine	46° Cosin	46° Sine	45° Cosin	45° Sine	′

TM684-156

′	0° Tang	0° Cotang	1° Tang	1° Cotang	2° Tang	2° Cotang	3° Tang	3° Cotang	′
0	.00000	Infinite.	.01746	57.2900	.03492	28.6363	.05241	19.0811	60
1	.00029	3437.75	.01775	56.3506	.03521	28.3994	.05270	18.9755	59
2	.00058	1718.87	.01804	55.4415	.03550	28.1664	.05299	18.8711	58
3	.00087	1145.92	.01833	54.5613	.03579	27.9372	.05328	18.7678	57
4	.00116	859.436	.01862	53.7086	.03609	27.7117	.05357	18.6656	56
5	.00145	687.549	.01891	52.8821	.03638	27.4899	.05387	18.5645	55
6	.00175	572.957	.01920	52.0807	.03667	27.2715	.05416	18.4645	54
7	.00204	491.106	.01949	51.3032	.03696	27.0566	.05445	18.3655	53
8	.00233	429.718	.01978	50.5485	.03725	26.8450	.05474	18.2677	52
9	.00262	381.971	.02007	49.8157	.03754	26.6367	.05503	18.1708	51
10	.00291	343.774	.02036	49.1039	.03783	26.4316	.05533	18.0750	50
11	.00320	312.521	.02066	48.4121	.03812	26.2296	.05562	17.9802	49
12	.00349	286.478	.02095	47.7395	.03842	26.0307	.05591	17.8863	48
13	.00378	264.441	.02124	47.0853	.03871	25.8348	.05620	17.7934	47
14	.00407	245.552	.02153	46.4489	.03900	25.6418	.05649	17.7015	46
15	.00436	229.182	.02182	45.8294	.03929	25.4517	.05678	17.6106	45
16	.00465	214.858	.02211	45.2261	.03958	25.2644	.05708	17.5205	44
17	.00495	202.219	.02240	44.6386	.03987	25.0798	.05737	17.4314	43
18	.00524	190.984	.02269	44.0661	.04016	24.8978	.05766	17.3432	42
19	.00553	180.932	.02298	43.5081	.04046	24.7185	.05795	17.2558	41
20	.00582	171.885	.02328	42.9641	.04075	24.5418	.05824	17.1693	40
21	.00611	163.700	.02357	42.4335	.04104	24.3675	.05854	17.0837	39
22	.00640	156.259	.02386	41.9158	.04133	24.1957	.05883	16.9990	38
23	.00669	149.465	.02415	41.4106	.04162	24.0263	.05912	16.9150	37
24	.00698	143.237	.02444	40.9174	.04191	23.8593	.05941	16.8319	36
25	.00727	137.507	.02478	40.4358	.04220	23.6945	.05970	16.7496	35
26	.00756	132.219	.02502	39.9655	.04250	23.5321	.05999	16.6681	34
27	.00785	127.321	.02531	39.5059	.04279	23.3718	.06029	16.5874	33
28	.00814	122.774	.02560	39.0568	.04308	23.2137	.06058	16.5075	32
29	.00844	118.540	.02589	38.6177	.04337	23.0577	.06087	16.4283	31
30	.00873	114.589	.02619	38.1885	.04366	22.9038	.06116	16.3499	30
31	.00902	110.892	.02648	37.7686	.04395	22.7519	.06145	16.2722	29
32	.00931	107.426	.02677	37.3579	.04424	22.6020	.06175	16.1952	28
33	.00960	104.171	.02706	36.9560	.04454	22.4541	.06204	16.1190	27
34	.00989	101.107	.02735	36.5627	.04483	22.3081	.06233	16.0435	26
35	.01018	98.2179	.02764	36.1776	.04512	22.1640	.06262	15.9687	25
36	.01047	95.4895	.02793	35.8006	.04541	22.0217	.06291	15.8945	24
37	.01076	92.9085	.02822	35.4313	.04570	21.8813	.06321	15.8211	23
38	.01105	90.4633	.02851	35.0695	.04599	21.7426	.06350	15.7483	22
39	.01135	88.1436	.02881	34.7151	.04628	21.6056	.06379	15.6762	21
40	.01164	85.9398	.02910	34.3678	.04658	21.4704	.06408	15.6048	20
41	.01193	83.8435	.02939	34.0273	.04687	21.3369	.06437	15.5340	19
42	.01222	81.8470	.02968	33.6935	.04716	21.2049	.06467	15.4638	18
43	.01251	79.9434	.02997	33.3662	.04745	21.0747	.06496	15.3943	17
44	.01280	78.1263	.03026	33.0452	.04774	20.9460	.06525	15.3254	16
45	.01309	76.3900	.03055	32.7303	.04803	20.8188	.06554	15.2571	15
46	.01338	74.7292	.03084	32.4213	.04832	20.6932	.06584	15.1893	14
47	.01367	73.1390	.03114	32.1181	.04862	20.5691	.06613	15.1222	13
48	.01396	71.6151	.03143	31.8205	.04891	20.4465	.06642	15.0557	12
49	.01425	70.1533	.03172	31.5284	.04920	20.3253	.06671	14.9898	11
50	.01455	68.7501	.03201	31.2416	.04949	20.2056	.06700	14.9244	10
51	.01484	67.4019	.03230	30.9599	.04978	20.0872	.06730	14.8596	9
52	.01513	66.1055	.03259	30.6823	.05007	19.9702	.06759	14.7954	8
53	.01542	64.8580	.03288	30.4116	.05037	19.8546	.06788	14.7317	7
54	.01571	63.6567	.03317	30.1446	.05066	19.7403	.06817	14.6685	6
55	.01600	62.4992	.03346	29.8823	.05095	19.6273	.06847	14.6059	5
56	.01629	61.3829	.03376	29.6245	.05124	19.5156	.06876	14.5438	4
57	.01658	60.3058	.03405	29.3711	.05153	19.4051	.06905	14.4823	3
58	.01687	59.2659	.03434	29.1220	.05182	19.2959	.06934	14.4212	2
59	.01716	58.2612	.03463	28.8771	.05212	19.1879	.06963	14.3607	1
60	.01746	57.2900	.03492	28.6363	.05241	19.0811	.06993	14.3007	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	89°		88°		87°		86°		

TM684-157

′	4° Tang	4° Cotang	5° Tang	5° Cotang	6° Tang	6° Cotang	7° Tang	7° Cotang	′
0	.06993	14.3007	.08749	11.4301	.10510	9.51436	.12278	8.14435	60
1	.07022	14.2411	.08778	11.3919	.10540	9.48781	.12308	8.12481	59
2	.07051	14.1821	.08807	11.3540	.10569	9.46141	.12338	8.10536	58
3	.07080	14.1235	.08837	11.3163	.10599	9.43515	.12367	8.08600	57
4	.07110	14.0655	.08866	11.2789	.10628	9.40904	.12397	8.06674	56
5	.07139	14.0079	.08895	11.2417	.10657	9.38307	.12426	8.04756	55
6	.07168	13.9507	.08925	11.2048	.10687	9.35724	.12456	8.02848	54
7	.07197	13.8940	.08954	11.1681	.10716	9.33154	.12485	8.00948	53
8	.07227	13.8378	.08983	11.1316	.10746	9.30599	.12515	7.99058	52
9	.07256	13.7821	.09013	11.0954	.10775	9.28058	.12544	7.97176	51
10	.07285	13.7267	.09042	11.0594	.10805	9.25530	.12574	7.95302	50
11	.07314	13.6719	.09071	11.0237	.10834	9.23016	.12603	7.93438	49
12	.07344	13.6174	.09101	10.9882	.10863	9.20516	.12633	7.91582	48
13	.07373	13.5634	.09130	10.9529	.10893	9.18028	.12662	7.89734	47
14	.07402	13.5098	.09159	10.9178	.10922	9.15554	.12692	7.87895	46
15	.07431	13.4566	.09189	10.8829	.10952	9.13093	.12722	7.86064	45
16	.07461	13.4039	.09218	10.8483	.10981	9.10646	.12751	7.84242	44
17	.07490	13.3515	.09247	10.8139	.11011	9.08211	.12781	7.82428	43
18	.07519	13.2996	.09277	10.7797	.11040	9.05789	.12810	7.80622	42
19	.07548	13.2480	.09306	10.7457	.11070	9.03379	.12840	7.78825	41
20	.07578	13.1969	.09335	10.7119	.11099	9.00983	.12869	7.77035	40
21	.07607	13.1461	.09365	10.6783	.11128	8.98598	.12899	7.75254	39
22	.07636	13.0958	.09394	10.6450	.11158	8.96227	.12929	7.73480	38
23	.07665	13.0458	.09423	10.6118	.11187	8.93867	.12958	7.71715	37
24	.07695	12.9962	.09453	10.5789	.11217	8.91520	.12988	7.69957	36
25	.07724	12.9469	.09482	10.5462	.11246	8.89185	.13017	7.68208	35
26	.07753	12.8981	.09511	10.5136	.11276	8.86862	.13047	7.66466	34
27	.07782	12.8496	.09541	10.4813	.11305	8.84551	.13076	7.64732	33
38	.07812	12.8014	.09570	10.4491	.11335	8.82252	.13106	7.63005	32
29	.07841	12.7536	.09600	10.4172	.11364	8.79964	.13136	7.61287	31
30	.07870	12.7062	.09629	10.3854	.11394	8.77689	.13165	7.59575	30
31	.07899	12.6591	.09658	10.3538	.11423	8.75425	.13195	7.57872	29
32	.07929	12.6124	.09688	10.3224	.11452	8.73172	.13224	7.56176	28
33	.07958	12.5660	.09717	10.2913	.11482	8.70931	.13254	7.54487	27
34	.07987	12.5199	.09746	10.2602	.11511	8.68701	.13284	7.52806	26
35	.08017	12.4742	.09776	10.2294	.11541	8.66482	.13313	7.51132	25
36	.08046	12.4288	.09805	10.1988	.11570	8.64275	.13343	7.49465	24
37	.08075	12.3838	.09834	10.1683	.11600	8.62078	.13372	7.47806	23
38	.08104	12.3390	.09864	10.1381	.11629	8.59893	.13402	7.46154	22
39	.08134	12.2946	.09893	10.1080	.11659	8.57718	.13432	7.44509	21
40	.08163	12.2505	.09923	10.0780	.11688	8.55555	.13461	7.42871	20
41	.08192	12.2067	.09952	10.0483	.11718	8.53402	.13491	7.41240	19
42	.08221	12.1632	.09981	10.0187	.11747	8.51259	.13521	7.39616	18
43	.08251	12.1201	.10011	9.98931	.11777	8.49128	.13550	7.37999	17
44	.08280	12.0772	.10040	9.96007	.11806	8.47007	.13580	7.36389	16
45	.08309	12.0346	.10069	9.93101	.11836	8.44896	.13609	7.34786	15
46	.08339	11.9923	.10099	9.90211	.11865	8.42795	.13639	7.33190	14
47	.08368	11.9504	.10128	9.87338	.11895	8.40705	.13669	7.31600	13
48	.08397	11.9087	.10158	9.84482	.11924	8.38625	.13698	7.30018	12
49	.08427	11.8673	.10187	9.81641	.11954	8.36555	.13728	7.28442	11
50	.08456	11.8262	.10216	9.78817	.11983	8.34496	.13758	7.26873	10
51	.08485	11.7853	.10246	9.76009	.12013	8.32446	.13787	7.25310	9
52	.08514	11.7448	.10275	9.73217	.12042	8.30406	.13817	7.23754	8
53	.08544	11.7045	.10305	9.70441	.12072	8.28376	.13846	7.22204	7
54	.08573	11.6645	.10334	9.67680	.12101	8.26355	.13876	7.20661	6
55	.08602	11.6248	.10363	9.64935	.12131	8.24345	.13906	7.19125	5
56	.08632	11.5853	.10393	9.62205	.12160	8.22344	.13935	7.17594	4
57	.08661	11.5461	.10422	9.59490	.12190	8.20352	.13965	7.16071	3
58	.08690	11.5072	.10452	9.56791	.12219	8.18370	.13995	7.14553	2
59	.08720	11.4685	.10481	9.54106	.12249	8.16398	.14024	7.13042	1
60	.08749	11.4301	.10510	9.51436	.12278	8.14435	.14054	7.11537	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	85°		84°		83°		82°		

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′	8° Tang	8° Cotang	9° Tang	9° Cotang	10° Tang	10° Cotang	11° Tang	11° Cotang	′
0	.14054	7.11537	.15838	6.31375	.17633	5.67128	.19438	5.14455	60
1	.14084	7.10038	.15868	6.30189	.17663	5.66165	.19468	5.13658	59
2	.14113	7.08546	.15898	6.29007	.17693	5.65205	.19498	5.12862	58
3	.14143	7.07059	.15928	6.27829	.17723	5.64248	.19529	5.12069	57
4	.14173	7.05579	.15958	6.26655	.17753	5.63295	.19559	5.11279	56
5	.14202	7.04105	.15988	6.25486	.17783	5.62344	.19589	5.10490	55
6	.14232	7.02637	.16017	6.24321	.17813	5.61397	.19619	5.09704	54
7	.14262	7.01174	.16047	6.23160	.17843	5.60452	.19649	5.08921	53
8	.14291	6.99718	.16077	6.22003	.17873	5.59511	.19680	5.08139	52
9	.14321	6.98268	.16107	6.20851	.17903	5.58573	.19710	5.07360	51
10	.14351	6.96823	.16137	6.19703	.17933	5.57638	.19740	5.06584	50
11	.14381	6.95385	.16167	6.18559	.17963	5.56706	.19770	5.05809	49
12	.14410	6.93952	.16196	6.17419	.17993	5.55777	.19801	5.05037	48
13	.14440	6.92525	.16226	6.16283	.18023	5.54851	.19831	5.04267	47
14	.14470	6.91104	.16256	6.15151	.18053	5.53927	.19861	5.03499	46
15	.14499	6.89688	.16286	6.14023	.18083	5.53007	.19891	5.02734	45
16	.14529	6.88278	.16316	6.12899	.18113	5.52090	.19921	5.01971	44
17	.14559	6.86874	.16346	6.11779	.18143	5.51176	.19952	5.01210	43
18	.14588	6.85475	.16376	6.10664	.18173	5.50264	.19982	5.00451	42
19	.14618	6.84082	.16405	6.09552	.18203	5.49356	.20012	4.99695	41
20	.14648	6.82694	.16435	6.08444	.18233	5.48451	.20042	4.98940	40
21	.14678	6.81312	.16465	6.07340	.18263	5.47548	.20073	4.98188	39
22	.14707	6.79936	.16495	6.06240	.18293	5.46648	.20103	4.97438	38
23	.14737	6.78564	.16525	6.05143	.18323	5.45751	.20133	4.96690	37
24	.14767	6.77199	.16555	6.04051	.18353	5.44857	.20164	4.95945	36
25	.14796	6.75838	.16585	6.02962	.18383	5.43966	.20194	4.95201	35
26	.14826	6.74483	.16615	6.01878	.18414	5.43077	.20224	4.94460	34
27	.14856	6.73133	.16645	6.00797	.18444	5.42192	.20254	4.93721	33
28	.14886	6.71789	.16674	5.99720	.18474	5.41309	.20285	4.92984	32
29	.14915	6.70450	.16704	5.98646	.18504	5.40429	.20315	4.92249	31
30	.14945	6.69116	.16734	5.97576	.18534	5.39552	.20345	4.91516	30
31	.14975	6.67787	.16764	5.96510	.18564	5.38677	.20376	4.90785	29
32	.15005	6.66463	.16794	5.95448	.18594	5.37805	.20406	4.90056	28
33	.15034	6.65144	.16824	5.94390	.18624	5.36936	.20436	4.89330	27
34	.15064	6.63831	.16854	5.93335	.18654	5.36070	.20466	4.88605	26
35	.15094	6.62523	.16884	5.92283	.18684	5.35206	.20497	4.87882	25
36	.15124	6.61219	.16914	5.91235	.18714	5.34345	.20527	4.87162	24
37	.15153	6.59921	.16944	5.90191	.18745	5.33487	.20557	4.86444	23
38	.15183	6.58627	.16974	5.89151	.18775	5.32631	.20588	4.85727	22
39	.15213	6.57339	.17004	5.88114	.18805	5.31778	.20618	4.85013	21
40	.15243	6.56055	.17033	5.87080	.18835	5.30928	.20648	4.84300	20
41	.15272	6.54777	.17063	5.86051	.18865	5.30080	.20679	4.83590	19
42	.15302	6.53503	.17093	5.85024	.18895	5.29235	.20709	4.82882	18
43	.15332	6.52234	.17123	5.84001	.18925	5.28393	.20739	4.82175	17
44	.15362	6.50970	.17153	5.82982	.18955	5.27553	.20770	4.81471	16
45	.15391	6.49710	.17183	5.81966	.18986	5.26715	.20800	4.80769	15
46	.15421	6.48456	.17213	5.80953	.19016	5.25880	.20830	4.80068	14
47	.15451	6.47206	.17243	5.79944	.19046	5.25048	.20861	4.79370	13
48	.15481	6.45961	.17273	5.78938	.19076	5.24218	.20891	4.78673	12
49	.15511	6.44720	.17303	5.77936	.19106	5.23391	.20921	4.77978	11
50	.15540	6.43484	.17333	5.76937	.19136	5.22566	.20952	4.77286	10
51	.15570	6.42253	.17363	5.75941	.19166	5.21744	.20982	4.76595	9
52	.15600	6.41026	.17393	5.74949	.19197	5.20925	.21013	4.75906	8
53	.15630	6.39804	.17423	5.73960	.19227	5.20107	.21043	4.75219	7
54	.15660	6.38587	.17453	5.72974	.19257	5.19293	.21073	4.74534	6
55	.15689	6.37374	.17483	5.71992	.19287	5.18480	.21104	4.73851	5
56	.15719	6.36165	.17513	5.71013	.19317	5.17671	.21134	4.73170	4
57	.15749	6.34961	.17543	5.70037	.19347	5.16863	.21164	4.72490	3
58	.15779	6.33761	.17573	5.69064	.19378	5.16058	.21195	4.71813	2
59	.15809	6.32566	.17603	5.68094	.19408	5.15256	.21225	4.71137	1
60	.15838	6.31375	.17633	5.67128	.19438	5.14455	.21256	4.70463	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	81°		80°		79°		78°		

TM884-159

′	12° Tang	12° Cotang	13° Tang	13° Cotang	14° Tang	14° Cotang	15° Tang	15° Cotang	′
0	.21256	4.70463	.23087	4.33148	.24933	4.01078	.26795	3.73205	60
1	.21286	4.69791	.23117	4.32573	.24964	4.00582	.26826	3.72771	59
2	.21316	4.69121	.23148	4.32001	.24995	4.00086	.26857	3.72338	58
3	.21347	4.68452	.23179	4.31430	.25026	3.99592	.26888	3.71907	57
4	.21377	4.67786	.23209	4.30860	.25056	3.99099	.26920	3.71476	56
5	.21408	4.67121	.23240	4.30291	.25087	3.98607	.26951	3.71046	55
6	.21438	4.66458	.23271	4.29724	.25118	3.98117	.26982	3.70616	54
7	.21469	4.65797	.23301	4.29159	.25149	3.97627	.27013	3.70188	53
8	.21499	4.65138	.23332	4.28595	.25180	3.97139	.27044	3.69761	52
9	.21529	4.64480	.23363	4.28032	.25211	3.96651	.27076	3.69335	51
10	.21560	4.63825	.23393	4.27471	.25242	3.96165	.27107	3.68909	50
11	.21590	4.63171	.23424	4.26911	.25273	3.95680	.27138	3.68485	49
12	.21621	4.62518	.23455	4.26352	.25304	3.95196	.27169	3.68061	48
13	.21651	4.61868	.23485	4.25795	.25335	3.94713	.27201	3.67638	47
14	.21682	4.61219	.23516	4.25239	.25366	3.94232	.27232	3.67217	46
15	.21712	4.60572	.23547	4.24685	.25397	3.93751	.27263	3.66796	45
16	.21743	4.59927	.23578	4.24132	.25428	3.93271	.27294	3.66376	44
17	.21773	4.59283	.23608	4.23580	.25459	3.92793	.27326	3.65957	43
18	.21804	4.58641	.23639	4.23030	.25490	3.92316	.27357	3.65538	42
19	.21834	4.58001	.23670	4.22481	.25521	3.91839	.27388	3.65121	41
20	.21864	4.57363	.23700	4.21933	.25552	3.91364	.27419	3.64705	40
21	.21895	4.56726	.23731	4.21387	.25583	3.90890	.27451	3.64289	39
22	.21925	4.56091	.23762	4.20842	.25614	3.90417	.27482	3.63874	38
23	.21956	4.55458	.23793	4.20298	.25645	3.89945	.27513	3.63461	37
24	.21986	4.54826	.23823	4.19756	.25676	3.89474	.27545	3.63048	36
25	.22017	4.54196	.23854	4.19215	.25707	3.89004	.27576	3.62636	35
26	.22047	4.53568	.23885	4.18675	.25738	3.88536	.27607	3.62224	34
27	.22078	4.52941	.23916	4.18137	.25769	3.88068	.27638	3.61814	33
28	.22108	4.52316	.23946	4.17600	.25800	3.87601	.27670	3.61405	32
29	.22139	4.51693	.23977	4.17064	.25831	3.87136	.27701	3.60996	31
30	.22169	4.51071	.24008	4.16530	.25862	3.86671	.27732	3.60588	30
31	.22200	4.50451	.24039	4.15997	.25893	3.86208	.27764	3.60181	29
32	.22231	4.49832	.24069	4.15465	.25924	3.85745	.27795	3.59775	28
33	.22261	4.49215	.24100	4.14934	.25955	3.85284	.27826	3.59370	27
34	.22292	4.48600	.24131	4.14405	.25986	3.84824	.27858	3.58966	26
35	.22322	4.47986	.24162	4.13877	.26017	3.84364	.27889	3.58562	25
36	.22353	4.47374	.24193	4.13350	.26048	3.83906	.27921	3.58160	24
37	.22383	4.46764	.24223	4.12825	.26079	3.83449	.27952	3.57758	23
38	.22414	4.46155	.24254	4.12301	.26110	3.82992	.27983	3.57357	22
39	.22444	4.45548	.24285	4.11778	.26141	3.82537	.28015	3.56957	21
40	.22475	4.44942	.24316	4.11256	.26172	3.82083	.28046	3.56557	20
41	.22505	4.44338	.24347	4.10736	.26203	3.81630	.28077	3.56159	19
42	.22536	4.43735	.24377	4.10216	.26235	3.81177	.28109	3.55761	18
43	.22567	4.43134	.24408	4.09699	.26266	3.80726	.28140	3.55364	17
44	.22597	4.42534	.24439	4.09182	.26297	3.80276	.28172	3.54968	16
45	.22628	4.41936	.24470	4.08666	.26328	3.79827	.28203	3.54573	15
46	.22658	4.41340	.24501	4.08152	.26359	3.79378	.28234	3.54179	14
47	.22689	4.40745	.24532	4.07639	.26390	3.78931	.28266	3.53785	13
48	.22719	4.40152	.24562	4.07127	.26421	3.78485	.28297	3.53393	12
49	.22750	4.39560	.24593	4.06616	.26452	3.78040	.28329	3.53001	11
50	.22781	4.38969	.24624	4.06107	.26483	3.77595	.28360	3.52609	10
51	.22811	4.38381	.24655	4.05599	.26515	3.77152	.28391	3.52219	9
52	.22842	4.37793	.24686	4.05092	.26546	3.76709	.28423	3.51829	8
53	.22872	4.37207	.24717	4.04586	.26577	3.76268	.28454	3.51441	7
54	.22903	4.36623	.24747	4.04081	.26608	3.75828	.28486	3.51053	6
55	.22934	4.36040	.24778	4.03578	.26639	3.75388	.28517	3.50666	5
56	.22964	4.35459	.24809	4.03075	.26670	3.74950	.28549	3.50279	4
57	.22995	4.34879	.24840	4.02574	.26701	3.74512	.28580	3.49894	3
58	.23026	4.34300	.24871	4.02074	.26733	3.74075	.28612	3.49509	2
59	.23056	4.33723	.24902	4.01576	.26764	3.73640	.28643	3.49125	1
60	.23087	4.33148	.24933	4.01078	.26795	3.73205	.28675	3.48741	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	77°		76°		75°		74°		

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′	16° Tang	16° Cotang	17° Tang	17° Cotang	18° Tang	18° Cotang	19° Tang	19° Cotang	′
0	.28675	3.48741	.30573	3.27085	.32492	3.07768	.34433	2.90421	60
1	.28706	3.48359	.30605	3.26745	.32524	3.07464	.34465	2.90147	59
2	.28738	3.47977	.30637	3.26406	.32556	3.07160	.34498	2.89873	58
3	.28769	3.47596	.30669	3.26067	.32588	3.06857	.34530	2.89600	57
4	.28800	3.47216	.30700	3.25729	.32621	3.06554	.34563	2.89327	56
5	.28832	3.46837	.30732	3.25392	.32653	3.06252	.34596	2.89055	55
6	.28864	3.46458	.30764	3.25055	.32685	3.05950	.34628	2.88783	54
7	.28895	3.46080	.30796	3.24719	.32717	3.05649	.34661	2.88511	53
8	.28927	3.45703	.30828	3.24383	.32749	3.05349	.34693	2.88240	52
9	.28958	3.45327	.30860	3.24049	.32782	3.05049	.34726	2.87970	51
10	.28990	3.44951	.30891	3.23714	.32814	3.04749	.34758	2.87700	50
11	.29021	3.44576	.30923	3.23381	.32846	3.04450	.34791	2.87430	49
12	.29053	3.44202	.30955	3.23048	.32878	3.04152	.34824	2.87161	48
13	.29084	3.43829	.30987	3.22715	.32911	3.03854	.34856	2.86892	47
14	.29116	3.43456	.31019	3.22384	.32943	3.03556	.34889	2.86624	46
15	.29147	3.43084	.31051	3.22053	.32975	3.03260	.34922	2.86356	45
16	.29179	3.42713	.31083	3.21722	.33007	3.02963	.34954	2.86089	44
17	.29210	3.42343	.31115	3.21392	.33040	3.02667	.34987	2.85822	43
18	.29242	3.41973	.31147	3.21063	.33072	3.02372	.35019	2.85555	42
19	.29274	3.41604	.31178	3.20734	.33104	3.02077	.35052	2.85289	41
20	.29305	3.41236	.31210	3.20406	.33136	3.01783	.35085	2.85023	40
21	.29337	3.40869	.31242	3.20079	.33169	3.01489	.35117	2.84758	39
22	.29368	3.40502	.31274	3.19752	.33201	3.01196	.35150	2.84494	38
23	.29400	3.40136	.31306	3.19426	.33233	3.00903	.35183	2.84229	37
24	.29432	3.39771	.31338	3.19100	.33266	3.00611	.35216	2.83965	36
25	.29463	3.39406	.31370	3.18775	.33298	3.00319	.35248	2.83702	35
26	.29495	3.39042	.31402	3.18451	.33330	3.00028	.35281	2.83439	34
27	.29526	3.38679	.31434	3.18127	.33363	2.99738	.35314	2.83176	33
28	.29558	3.38317	.31466	3.17804	.33395	2.99447	.35346	2.82914	32
29	.29590	3.37955	.31498	3.17481	.33427	2.99158	.35379	2.82653	31
30	.29621	3.37594	.31530	3.17159	.33460	2.98868	.35412	2.82391	30
31	.29653	3.37234	.31562	3.16838	.33492	2.98580	.35445	2.82130	29
32	.29685	3.36875	.31594	3.16517	.33524	2.98292	.35477	2.81870	28
33	.29716	3.36516	.31626	3.16197	.33557	2.98004	.35510	2.81610	27
34	.29748	3.36158	.31658	3.15877	.33589	2.97717	.35543	2.81350	26
35	.29780	3.35800	.31690	3.15558	.33621	2.97430	.35576	2.81091	25
36	.29811	3.35443	.31722	3.15240	.33654	2.97144	.35608	2.80833	24
37	.29843	3.35087	.31754	3.14922	33686	2.96858	.35641	2.80574	23
38	.29875	3.34732	.31786	3.14605	.33718	2.96573	.35674	2.80316	22
39	.29906	3.34377	.31818	3.14288	.33751	2.96288	.35707	2.80059	21
40	.29938	3.34023	.31850	3.13972	.33783	2.96004	.35740	2.79802	20
41	.29970	3.33670	.31882	3.13656	.33816	2.95721	.35772	2.79545	19
42	.30001	3.33317	.31914	3.13341	.33848	2.95437	.35805	2.79289	18
43	.30033	3.32965	.31946	3.13027	.33881	2.95155	.35838	2.79033	17
44	.30065	3.32614	.31978	3.12713	.33913	2.94872	.35871	2.78778	16
45	.30097	3.32264	.32010	3.12400	.33945	2.94590	.35904	2.78523	15
46	.30128	3.31914	.32042	3.12087	.33978	2.94309	.35937	2.78269	14
47	.30160	3.31565	.32074	3.11775	.34010	2.94028	.35969	2.78014	13
48	.30192	3.31216	.32106	3.11464	.34043	2.93748	.36002	2.77761	12
49	.30224	3.30868	.32139	3.11153	.34075	2.93468	36035	2.77507	11
50	.30255	3.30521	.32171	3.10842	.34108	2.93189	.36068	2.77254	10
51	.30287	3.30174	.32203	3.10532	.34140	2.92910	.36101	2.77002	9
52	.30319	3.29829	.32235	3.10223	.34173	2.92632	.36134	2.76750	8
53	.30351	3.29483	.32267	3.09914	.34205	2.92354	.36167	2.76498	7
54	.30382	3.29139	.32299	3.09606	.34238	2.92076	.36199	2.76247	6
55	.30414	3.28795	.32331	3.09298	.34270	2.91799	.36232	2.75996	5
56	.30446	3.28452	.32363	3.08991	.34303	2.91523	.36265	2.75746	4
57	.30478	3.28109	.32396	3.08685	.34335	2.91246	.36298	2.75496	3
58	.30509	3.27767	.32428	3.08379	.34368	2.90971	.36331	2.75246	2
59	.30541	3.27426	.32460	3.08073	.34400	2.90696	.36364	2.74997	1
60	.30573	3.27085	.32492	3.07768	.34433	2.90421	.36397	2.74748	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	73°		72°		71°		70°		

TM684-161

′	20° Tang	20° Cotang	21° Tang	21° Cotang	22° Tang	22° Cotang	23° Tang	23° Cotang	′
0	.36397	2.74748	.38386	2.60509	.40403	2.47509	.42447	2.35585	60
1	.36430	2.74499	.38420	2.60283	.40436	2.47302	.42482	2.35395	59
2	.36463	2.74251	.38453	2.60057	.40470	2.47095	.42516	2.35205	58
3	.36496	2.74004	.38487	2.59831	.40504	2.46888	.42551	2.35015	57
4	.36529	2.73756	.38520	2.59606	.40538	2.46682	.42585	2.34825	56
5	.36562	2.73509	.38553	2.59381	.40572	2.46476	.42619	2.34636	55
6	.36595	2.73263	.38587	2.59156	.40606	2.46270	.42654	2.34447	54
7	.36628	2.73017	.38620	2.58932	.40640	2.46065	.42688	2.34258	53
8	.36661	2.72771	.38654	2.58708	.40674	2.45860	.42722	2.34069	52
9	.36694	2.72526	.38687	2.58484	.40707	2.45655	.42757	2.33881	51
10	.36727	2.72281	.38721	2.58261	.40741	2.45451	.42791	2.33693	50
11	.36760	2.72036	.38754	2.58038	.40775	2.45246	.42826	2.33505	49
12	.36793	2.71792	.38787	2.57815	.40809	2.45043	.42860	2.33317	48
13	.36826	2.71548	.38821	2.57593	.40843	2.44839	.42894	2.33130	47
14	.36859	2.71305	.38854	2.57371	.40877	2.44636	.42929	2.32943	46
15	36892	2.71062	.38888	2.57150	.40911	2.44433	.42963	2.32756	45
16	.36925	2.70819	.38921	2.56928	.40945	2.44230	.42998	2.32570	44
17	.36958	2.70577	.38955	2.56707	.40979	2.44027	.43032	2.32383	43
18	.36991	2.70335	.38988	2.56487	.41013	2.43825	.43067	2.32197	42
19	.37024	2.70094	.39022	2.56266	.41047	2.43623	.43101	2.32012	41
20	.37057	2.69853	.39055	2.56046	.41081	2.43422	.43136	2.31826	40
21	.37090	2.69612	.39089	2.55827	.41115	2.43220	.43170	2.31641	39
22	.37124	2.69371	.39122	2.55608	.41149	2.43019	.43205	2.31456	38
23	.37157	2.69131	.39156	2.55389	.41183	2.42819	.43239	2.31271	37
24	.37190	2.68892	.39190	2.55170	.41217	2.42618	.43274	2.31086	36
25	.37223	2.68653	.39223	2.54952	.41251	2.42418	.43308	2.30902	35
26	.37256	2.68414	.39257	2.54734	.41285	2.42218	.43343	2.30718	34
27	.37289	2.68175	.39290	2.54516	.41319	2.42019	.43378	2.30534	33
28	.37322	2.67937	.39324	2.54299	.41353	2.41819	.43412	2.30351	32
29	.37355	2.67700	.39357	2.54082	.41387	2.41620	.43447	2.30167	31
30	.37388	2.67462	.39391	2.53865	.41421	2.41421	.43481	2.29984	30
31	.37422	2.67225	.39425	2.53648	.41455	2.41223	.43516	2.29801	29
32	.37455	2.66989	.39458	2.53432	.41490	2.41025	.43550	2.29619	28
33	.37488	2.66752	.39492	2.53217	.41524	2.40827	.43585	2.29437	27
34	.37521	2.66516	.39526	2.53001	.41558	2.40629	.43620	2.29254	26
35	.37554	2.66281	.39559	2.52786	.41592	2.40432	.43654	2.29073	25
36	.37588	2.66046	.39593	2.52571	.41626	2.40235	.43689	2.28891	24
37	.37621	2.65811	.39626	2.52357	.41660	2.40038	.43724	2.28710	23
38	.37654	2.65576	.39660	2.52142	.41694	2.39841	.43758	2.28528	22
39	.37687	2.65342	.39694	2.51929	.41728	2.39645	.43793	2.28348	21
40	.37720	2.65109	.39727	2.51715	.41763	2.39449	.43828	2.28167	20
41	.37754	2.64875	.39761	2.51502	.41797	2.39253	.43862	2.27987	19
42	.37787	2.64642	.39795	2.51289	.41831	2.39058	.43897	2.27806	18
43	.37820	2.64410	.39829	2.51076	.41865	2.38862	.43932	2.27626	17
44	.37853	2.64177	.39862	2.50864	.41899	2.38668	.43966	2.27447	16
45	.37887	2.63945	.39896	2.50652	.41933	2.38473	.44001	2.27267	15
46	.37920	2.63714	.39930	2.50440	.41968	2.38279	.44036	2.27088	14
47	.37953	2.63483	.39963	2.50229	.42002	2.38084	.44071	2.26909	13
48	.37986	2.63252	.39997	2.50018	.42036	2.37891	.44105	2.26730	12
49	.38020	2.63021	.40031	2.49807	.42070	2.37697	.44140	2.26552	11
50	.38053	2.62791	.40065	2.49597	.42105	2.37504	.44175	2.26374	10
51	.38086	2.62561	.40098	2.49386	.42139	2.37311	.44210	2.26196	9
52	.38120	2.62332	.40132	2.49177	.42173	2.37118	.44244	2.26018	8
53	.38153	.262103	.40166	2.48967	.42207	2.36925	.44279	2.25840	7
54	.38186	2.61874	.40200	2.48758	.42242	2.36733	.44314	2.25663	6
55	.38220	2.61646	.40234	2.48549	.42276	2.36541	.44349	2.25486	5
56	.38253	2.61418	.40267	2.48340	.42310	2.36349	.44384	2.25309	4
57	.38286	2.61190	.40301	2.48132	.42345	2.36158	.44418	2.25132	3
58	.38320	2.60963	.40335	2.47924	.42379	2.35967	.44453	2.24956	2
59	.38353	2.60736	.40369	2.47716	.42413	2.35776	.44488	2.24780	1
60	.38386	2.60509	.40403	2.47509	.42447	2.35585	.44523	2.24604	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	69°		68°		67°		66°		

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′	24° Tang	24° Cotang	25° Tang	25° Cotang	26° Tang	26° Cotang	27° Tang	27° Cotang	′
0	.44523	2.24604	.46631	2.14451	.48773	2.05030	.50953	1.96261	60
1	.44558	2.24428	.46666	2.14288	.48809	2.04879	.50989	1.96120	59
2	.44593	2.24252	.46702	2.14125	.48845	2.04728	.51026	1.95979	58
3	.44627	2.24077	.46737	2.13963	.48881	2.04577	.51063	1.95838	57
4	.44662	2.23902	.46772	2.13801	.48917	2.04426	.51099	1.95698	56
5	.44697	2.23727	.46808	2.13639	.48953	2.04276	.51136	1.95557	55
6	.44732	2.23553	.46843	2.13477	.48989	2.04125	.51173	1.95417	54
7	.44767	2.23378	.46879	2.13316	.49026	2.03975	.51209	1.95277	53
8	.44802	2.23204	.46914	2.13154	.49062	2.03825	.51246	1.95137	52
9	.44837	2.23030	.46950	2.12993	.49098	2.03675	.51283	1.94997	51
10	.44872	2.22857	.46985	2.12832	.49134	2.03526	.51319	1.94858	50
11	.44907	2.22683	.47021	2.12671	.49170	2.03376	.51356	1.94718	49
12	.44942	2.22510	.47056	2.12511	.49206	2.03227	.51393	1.94579	48
13	.44977	2.22337	.47092	2.12350	.49242	2.03078	.51430	1.94440	47
14	.45012	2.22164	.47128	2.12190	.49278	2.02929	.51467	1.94301	46
15	.45047	2.21992	.47163	2.12030	.49315	2.02780	.51503	1.94162	45
16	.45082	2.21819	.47199	2.11871	.49351	2.02631	.51540	1.94023	44
17	.45117	2.21647	.47234	2.11711	.49387	2.02483	.51577	1.93885	43
18	.45152	2.21475	.47270	2.11552	.49423	2.02335	.51614	1.93746	42
19	.45187	2.21304	.47305	2.11392	.49459	2.02187	.51651	1.93608	41
20	.45222	2.21132	.47341	2.11233	.49495	2.02039	.51688	1.93470	40
21	.45257	2.20961	.47377	2.11075	.49532	2.01891	.51724	1.93332	39
22	.45292	2.20790	.47412	2.10916	.49568	2.01743	.51761	1.93195	38
23	.45327	2.20619	.47448	2.10758	.49604	2.01596	.51798	1.93057	37
24	.45362	2.20449	.47483	2.10600	.49640	2.01449	.51835	1.92920	36
25	.45397	2.20278	.47519	2.10442	.49677	2.01302	.51872	1.92782	35
26	.45432	2.20108	.47555	2.10284	.49713	2.01155	.51909	1.92645	34
27	.45467	2.19938	.47590	2.10126	.49749	2.01008	.51946	1.92508	33
28	.45502	2.19769	.47626	2.09969	.49786	2.00862	.51983	1.92371	32
29	.45537	2.19599	.47662	2.09811	.49822	2.00715	.52020	1.92235	31
30	.45573	2.19430	.47698	2.09654	.49858	2.00569	.52057	1.92098	30
31	.45608	2.19261	.47733	2.09498	.49894	2.00423	.52094	1.91962	29
32	.45643	2.19092	.47769	2.09341	.49931	2.00277	.52131	1.91826	28
33	.45678	2.18923	.47805	2.09184	.49967	2.00131	.52168	1.91690	27
34	.45713	2.18755	.47840	2.09028	.50004	1.99986	.52205	1.91554	26
35	.45748	2.18587	.47876	2.08872	.50040	1.99841	.52242	1.91418	25
36	.45784	2.18419	.47912	2.08716	.50076	1.99695	.52279	1.91282	24
37	.45819	2.18251	.47948	2.08560	.50113	1.99550	.52316	1.91147	23
38	.45854	2.18084	.47984	2.08405	.50149	1.99406	.52353	1.91012	22
39	.45889	2.17916	.48019	2.08250	.50185	1.99261	.52390	1.90876	21
40	.45924	2.17749	.48055	2.08094	.50222	1.99116	.52427	1.90741	20
41	.45960	2.17582	.48091	2.07939	.50258	1.98972	.52464	1.90607	19
42	.45995	2.17416	.48127	2.07785	.50295	1.98828	.52501	1.90472	18
43	.46030	2.17249	.48163	2.07630	.50331	1.98684	.52538	1.90337	17
44	.46065	2.17083	.48198	2.07476	.50368	1.98540	.52575	1.90203	16
45	.46101	2.16917	.48234	2.07321	.50404	1.98396	.52613	1.90069	15
46	.46136	2.16751	.48270	2.07167	.50441	1.98253	.52650	1.89935	14
47	.46171	2.16585	.48306	2.07014	.50477	1.98110	.52687	1.89801	13
48	.46206	2.16420	.48342	2.06860	.50514	1.97966	.52724	1.89667	12
49	.46242	2.16255	.48378	2.06706	.50550	1.97823	.52761	1.89533	11
50	.46277	2.16090	.48414	2.06553	.50587	1.97680	.52798	1.89400	10
51	.46312	2.15925	.48450	2.06400	.50623	1.97538	.52836	1.89266	9
52	.46348	2.15760	.48486	2.06247	.50660	1.97395	.52873	1.89133	8
53	.46383	2.15596	.48521	2.06094	.50696	1.97253	.52910	1.89000	7
54	.46418	2.15432	.48557	2.05942	.50733	1.97111	.52947	1.88867	6
55	.46454	2.15268	.48593	2.05790	.50769	1.96969	.52984	1.88734	5
56	.46489	2.15104	.48629	2.05637	.50806	1.96827	.53022	1.88602	4
57	.46525	2.14940	.48665	2.05485	.50843	1.96685	.53059	1.88469	3
58	.46560	2.14777	.48701	2.05333	.50879	1.96544	.53096	1.88337	2
59	.46595	2.14614	.48737	2.05182	.50916	1.96402	.53134	1.88205	1
60	.46631	2.14451	.48773	2.05030	.50953	1.96261	.53171	1.88073	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	65°		64°		63°		62°		

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′	28° Tang	28° Cotang	29° Tang	29° Cotang	30° Tang	30° Cotang	31° Tang	31° Cotang	′
0	.53171	1.88073	.55431	1.80405	.57735	1.73205	.60086	1.66428	60
1	.53208	1.87941	.55469	1.80281	.57774	1.73089	.60126	1.66318	59
2	.53246	1.87809	.55507	1.80158	.57813	1.72973	.60165	1.66209	58
3	.53283	1.87677	.55545	1.80034	.57851	1.72857	.60205	1.66099	57
4	.53320	1.87546	.55583	1.79911	.57890	1.72741	.60245	1.65990	56
5	.53358	1.87415	.55621	1.79788	.57929	1.72625	.60284	1.65881	55
6	.53395	1.87283	.55659	1.79665	.57968	1.72509	.60324	1.65772	54
7	.53432	1.87152	.55697	1.79542	.58007	1.72393	.60364	1.65663	53
8	.53470	1.87021	.55736	1.79419	.58046	1.72278	.60403	1.65554	52
9	.53507	1.86891	.55774	1.79296	.58085	1.72163	.60443	1.65445	51
10	.53545	1.86760	.55812	1.79174	.58124	1.72047	.60483	1.65337	50
11	.53582	1.86630	.55850	1.79051	.58162	1.71932	.60522	1.65228	49
12	.53620	1.86499	.55888	1.78929	.58201	1.71817	.60562	1.65120	48
13	.53657	1.86369	.55926	1.78807	.58240	1.71702	.60602	1.65011	47
14	.53694	1.86239	.55964	1.78685	.58279	1.71588	.60642	1.64903	46
15	.53732	1.86109	.56003	1.78563	.58318	1.71473	.60681	1.64795	45
16	.53769	1.85979	.56041	1.78441	.58357	1.71358	.60721	1.64687	44
17	.53807	1.85850	.56079	1.78319	.58396	1.71244	.60761	1.64579	43
18	.53844	1.85720	.56117	1.78198	.58435	1.71129	.60801	1.64471	42
19	.53882	1.85591	.56156	1.78077	.58474	1.71015	.60841	1.64363	41
20	.53920	1.85462	.56194	1.77955	.58513	1.70901	.60881	1.64256	40
21	.53957	1.85333	.56232	1.77834	.58552	1.70787	.60921	1.64148	39
22	.53995	1.85204	.56270	1.77713	.58591	1.70673	.60960	1.64041	38
23	.54032	1.85075	.56309	1.77592	.58631	1.70560	.61000	1.63934	37
24	.54070	1.84946	.56347	1.77471	.58670	1.70446	.61040	1.63826	36
25	.54107	1.84818	.56385	1.77351	.58709	1.70332	.61080	1.63719	35
26	.54145	1.84689	.56424	1.77230	.58748	1.70219	.61120	1.63612	34
27	.54183	1.84561	.56462	1.77110	.58787	1.70106	.61160	1.63505	33
28	.54220	1.84433	.56500	1.76990	.58826	1.69992	.61200	1.63398	32
29	.54258	1.84305	.56539	1.76869	.58865	1.69879	.61240	1.63292	31
30	.54296	1.84177	.56577	1.76749	.58904	1.69766	.61280	1.63185	30
31	.54333	1.84049	.56616	1.76630	.58944	1.69653	.61320	1.63079	29
32	.54371	1.83922	.56654	1.76510	.58983	1.69541	.61360	1.62972	28
33	.54409	1.83794	.56693	1.76390	.59022	1.69428	.61400	1.62866	27
34	.54446	1.83667	.56731	1.76271	.59061	1.69316	.61440	1.62760	26
35	.54484	1.83540	.56769	1.76151	.59101	1.69203	.61480	1.62654	25
36	.54522	1.83413	.56808	1.76032	.59140	1.69091	.61520	1.62548	24
37	.54560	1.83286	.56846	1.75913	.59179	1.68979	.61561	1.62442	23
38	.54597	1.83159	.56885	1.75794	.59218	1.68866	.61601	1.62336	22
39	.54635	1.83033	.56923	1.75675	.59258	1.68754	.61641	1.62230	21
40	.54673	1.82906	.56962	1.75556	.59297	1.68643	.61681	1.62125	20
41	.54711	1.82780	.57000	1.75437	.59336	1.68531	.61721	1.62019	19
42	.54748	1.82654	.57039	1.75319	.59376	1.68419	.61761	1.61914	18
43	.54786	1.82528	.57078	1.75200	.59415	1.68308	.61801	1.61808	17
44	.54824	1.82402	.57116	1.75082	.59454	1.68196	.61842	1.61703	16
45	.54862	1.82276	.57155	1.74964	.59494	1.68085	.61882	1.61598	15
46	.54900	1.82150	.57193	1.74846	.59533	1.67974	.61922	1.61493	14
47	.54938	1.82025	.57232	1.74728	.59573	1.67863	.61962	1.61388	13
48	.54975	1.81899	.57271	1.74610	.59612	1.67752	.62003	1.61283	12
49	.55013	1.81774	.57309	1.74492	.59651	1.67641	.62043	1.61179	11
50	.55051	1.81649	.57348	1.74375	.59691	1.67530	.62083	1.61074	10
51	.55089	1.81524	.57386	1.74257	.59730	1.67419	.62124	1.60970	9
52	.55127	1.81399	.57425	1.74140	.59770	1.67309	.62164	1.60865	8
53	.55165	1.81274	.57464	1.74022	.59809	1.67198	.62204	1.60761	7
54	.55203	1.81150	.57503	1.73905	.59849	1.67088	.62245	1.60657	6
55	.55241	1.81025	.57541	1.73788	.59888	1.66978	.62285	1.60553	5
56	.55279	1.80901	.57580	1.73671	.59928	1.66867	.62325	1.60449	4
57	.55317	1.80777	.57619	1.73555	.59967	1.66757	.62366	1.60345	3
58	.55355	1.80653	.57657	1.73438	.60007	1.66647	.62406	1.60241	2
59	.55393	1.80529	.57696	1.73321	.60046	1.66538	.62446	1.60137	1
60	.55431	1.80405	.57735	1.73205	.60086	1.66428	.62487	1.60033	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	61°		60°		59°		58°		

TM684-164

′	32° Tang	32° Cotang	33° Tang	33° Cotang	34° Tang	34° Cotang	35° Tang	35° Cotang	′
0	.62487	1.60033	.64941	1.53986	.67451	1.48256	.70021	1.42815	60
1	.62527	1.59930	.64982	1.53888	.67493	1.48163	.70064	1.42726	59
2	.62568	1.59826	.65023	1.53791	.67536	1.48070	.70107	1.42638	58
3	.62608	1.59723	.65065	1.53693	.67578	1.47977	.70151	1.42550	57
4	.62649	1.59620	.65106	1.53595	.67620	1.47885	.70194	1.42462	56
5	.62689	1.59517	.65148	1.53497	.67663	1.47792	.70238	1.42374	55
6	.62730	1.59414	.65189	1.53400	.67705	1.47699	.70281	1.42286	54
7	.62770	1.59311	.65231	1.53302	.67748	1.47607	.70325	1.42198	53
8	.62811	1.59208	.65272	1.53205	.67790	1.47514	.70368	1.42110	52
9	.62852	1.59105	.65314	1.53107	.67832	1.47422	.70412	1.42022	51
10	.62892	1.59002	.65355	1.53010	.67875	1.47330	.70455	1.41934	50
11	.62933	1.58900	.65397	1.52913	.67917	1.47238	.70499	1.41847	49
12	.62973	1.58797	.65438	1.52816	.67960	1.47146	.70542	1.41759	48
13	.63014	1.58695	.65480	1.52719	.68002	1.47053	.70586	1.41672	47
14	.63055	1.58593	.65521	1.52622	.68045	1.46962	.70629	1.41584	46
15	.63095	1.58490	.65563	1.52525	.68088	1.46870	.70673	1.41497	45
16	.63136	1.58388	.65604	1.52429	.68130	1.46778	.70717	1.41409	44
17	.63177	1.58286	.65646	1.52332	.68173	1.46686	.70760	1.41322	43
18	.63217	1.58184	.65688	1.52235	.68215	1.46595	.70804	1.41235	42
19	.63258	1.58083	.65729	1.52139	.68258	1.46503	.70848	1.41148	41
20	.63299	1.57981	.65771	1.52043	.68301	1.46411	.70891	1.41061	40
21	.63340	1.57879	.65813	1.51946	.68343	1.46320	.70935	1.40974	39
22	.63380	1.57778	.65854	1.51850	.68386	1.46229	.70979	1.40887	38
23	.63421	1.57676	.65896	1.51754	.68429	1.46137	.71023	1.40800	37
24	.63462	1.57575	.65938	1.51658	.68471	1.46046	.71066	1.40714	36
25	.63503	1.57474	.65980	1.51562	.68514	1.45955	.71110	1.40627	35
26	.63544	1.57372	.66021	1.51466	.68557	1.45864	.71154	1.40540	34
27	.63584	1.57271	.66063	1.51370	.68600	1.45773	.71198	1.40454	33
28	.63625	1.57170	.66105	1.51275	.68642	1.45682	.71242	1.40367	32
29	.63666	1.57069	.66147	1.51179	.68685	1.45592	.71285	1.40281	31
30	.63707	1.56969	.66189	1.51084	.68728	1.45501	.71329	1.40195	30
31	.63748	1.56868	.66230	1.50988	.68771	1.45410	.71373	1.40109	29
32	.63789	1.56767	.66272	1.50893	.68814	1.45320	.71417	1.40022	28
33	.63830	1.56667	.66314	1.50799	.68857	1.45229	.71461	1.39936	27
34	.63871	1.56566	.66356	1.50702	.68900	1.45138	.71505	1.39850	26
35	.63912	1.56466	.66398	1.50607	.68942	1.45049	.71549	1.39764	25
36	.63953	1.56366	.66440	1.50512	.68985	1.44958	.71593	1.39679	24
37	.63994	1.56265	.66482	1.50417	.69028	1.44868	.71637	1.39593	23
38	.64035	1.56165	.66524	1.50322	.69071	1.44778	.71681	1.39507	22
39	.64076	1.56065	.66566	1.50228	.69114	1.44688	.71725	1.39421	21
40	.64117	1.55966	.66608	1.50133	.69157	1.44598	.71769	1.39336	20
41	.64158	1.55866	.66650	1.50038	.69200	1.44508	.71813	1.39250	19
42	.64199	1.55766	.66692	1.49944	.69243	1.44418	.71857	1.39165	18
43	.64240	1.55666	.66734	1.49849	.69286	1.44329	.71901	1.39079	17
44	.64281	1.55567	.66776	1.49755	.69329	1.44239	.71946	1.38994	16
45	.64322	1.55467	.66818	1.49661	.69372	1.44149	.71990	1.38909	15
46	.64363	1.55368	.66860	1.49566	.69416	1.44060	.72034	1.38824	14
47	.64404	1.55269	.66902	1.49472	.69459	1.43970	.72078	1.38738	13
48	.64446	1.55170	.66944	1.49378	.69502	1.43881	.72122	1.38653	12
49	.64487	1.55071	.66986	1.49284	.69545	1.43792	.72166	1.38568	11
50	.64528	1.54972	.67028	1.49190	.69588	1.43703	.72211	1.38484	10
51	.64569	1.54873	.67071	1.49097	.69631	1.43614	.72255	1.38399	9
52	.64610	1.54774	.67113	1.49003	.69675	1.43525	.72299	1.38314	8
53	.64652	1.54675	.67155	1.48909	.69718	1.43436	.72344	1.38229	7
54	.64693	1.54576	.67197	1.48816	.69761	1.43347	.72388	1.38145	6
55	.64734	1.54478	.67239	1.48722	.69804	1.43258	.72432	1.38060	5
56	.64775	1.54379	.67282	1.48629	.69847	1.43169	.72477	1.37976	4
57	.64817	1.54281	.67324	1.48536	.69891	1.43080	.72521	1.37891	3
58	.64858	1.54183	.67366	1.48442	.69934	1.42992	.72565	1.37807	2
59	.64899	1.54085	.67409	1.48349	.69977	1.42903	.72610	1.37722	1
60	.64941	1.53986	.67451	1.48256	.70021	1.42815	.72654	1.37638	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	57°		56°		55°		54°		

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′	36° Tang	36° Cotang	37° Tang	37° Cotang	38° Tang	38° Cotang	39° Tang	39° Cotang	′
0	.72654	1.37638	.75355	1.32704	.78129	1.27994	.80978	1.23490	60
1	.72699	1.37554	.75401	1.32624	.78175	1.27917	.81027	1.23416	59
2	.72743	1.37470	.75447	1.32544	.78222	1.27841	.81075	1.23343	58
3	.72788	1.37386	.75492	1.32464	.78269	1.27764	.81123	1.23270	57
4	.72832	1.37302	.75538	1.32384	.78316	1.27688	.81171	1.23196	56
5	.72877	1.37218	.75584	1.32304	.78363	1.27611	.81220	1.23123	55
6	.72921	1.37134	.75629	1.32224	.78410	1.27535	.81268	1.23050	54
7	.72966	1.37050	.75675	1.32144	.78457	1.27458	.81316	1.22977	53
8	.73010	1.36967	.75721	1.32064	.78504	1.27382	.81364	1.22904	52
9	.73055	1.36883	.75767	1.31984	.78551	1.27306	.81413	1.22831	51
10	.73100	1.36800	.75812	1.31904	.78598	1.27230	.81461	1.22758	50
11	.73144	1.36716	.75858	1.31825	.78645	1.27153	.81510	1.22685	49
12	.73189	1.36633	.75904	1.31745	.78692	1.27077	.81558	1.22612	48
13	.73234	1.36549	.75950	1.31666	.78739	1.27001	.81606	1.22539	47
14	.73278	1.36466	.75996	1.31586	.78786	1.26925	.81655	1.22467	46
15	.73323	1.36383	.76042	1.31507	.78834	1.26849	.81703	1.22394	45
16	.73368	1.36300	.76088	1.31427	.78881	1.26774	.81752	1.22321	44
17	.73413	1.36217	.76134	1.31348	.78928	1.26698	.81800	1.22249	43
18	.73457	1.36133	.76180	1.31269	.78975	1.26622	.81849	1.22176	42
19	.73502	1.36051	.76226	1.31190	.79022	1.26546	.81898	1.22104	41
20	.73547	1.35968	.76272	1.31110	.79070	1.26471	.81946	1.22031	40
21	.73592	1.35885	.76318	1.31031	.79117	1.26395	.81995	1.21959	39
22	.73637	1.35802	.76364	1.30952	.79164	1.26319	.82044	1.21886	38
23	.73681	1.35719	.76410	1.30873	.79212	1.26244	.82092	1.21814	37
24	.73726	1.35637	.76456	1.30795	.79259	1.26169	.82141	1.21742	36
25	.73771	1.35554	.76502	1.30716	.79306	1.26093	.82190	1.21670	35
26	.73816	1.35472	.76548	1.30637	.79354	1.26018	.82238	1.21598	34
27	.73861	1.35389	.76594	1.30558	.79401	1.25943	.82287	1.21526	33
28	.73906	1.35307	.76640	1.30480	.79449	1.25867	.82336	1.21454	32
29	.73951	1.35224	.76686	1.30401	.79496	1.25792	.82385	1.21382	31
30	.73996	1.35142	.76733	1.30323	.79544	1.25717	.82434	1.21310	30
31	.74041	1.35060	.76779	1.30244	.79591	1.25642	.82483	1.21238	29
32	.74086	1.34978	.76825	1.30166	.79639	1.25567	.82531	1.21166	28
33	.74131	1.34896	.76871	1.30087	.79686	1.25492	.82580	1.21094	27
34	.74176	1.34814	.76918	1.30009	.79734	1.25417	.82629	1.21023	26
35	.74221	1.34732	.76964	1.29931	.79781	1.25343	.82678	1.20951	25
36	.74267	1.34650	.77010	1.29853	.79829	1.25268	.82727	1.20879	24
37	.74312	1.34568	.77057	1.29775	.79877	1.25193	.82776	1.20808	23
38	.74357	1.34487	.77103	1.29696	.79924	1.25118	.82825	1.20736	22
39	.74402	1.34405	.77149	1.29618	.79972	1.25044	.82874	1.20665	21
40	.74447	1.34323	.77196	1.29541	.80020	1.24969	.82923	1.20593	20
41	.74492	1.34242	.77242	1.29463	.80067	1.24895	.82972	1.20522	19
42	.74538	1.34160	.77289	1.29385	.80115	1.24820	.83022	1.20451	18
43	.74583	1.34079	.77335	1.29307	.80163	1.24746	.83071	1.20379	17
44	.74628	1.33998	.77382	1.29229	.80211	1.24672	.83120	1.20308	16
45	.74674	1.33916	.77428	1.29152	.80258	1.24597	.83169	1.20237	15
46	.74719	1.33835	.77475	1.29074	.80306	1.24523	.83218	1.20166	14
47	.74764	1.33754	.77521	1.28997	.80354	1.24449	.83268	1.20095	13
48	.74810	1.33673	.77568	1.28919	.80402	1.24375	.83317	1.20024	12
49	.74855	1.33592	.77615	1.28842	.80450	1.24301	.83366	1.19953	11
50	.74900	1.33511	.77661	1.28764	.80498	1.24227	.83415	1.19882	10
51	.74946	1.33430	.77708	1.28687	.80546	1.24153	.83465	1.19811	9
52	.74991	1.33349	.77754	1.28610	.80594	1.24079	.83514	1.19740	8
53	.75037	1.33268	.77801	1.28533	.80642	1.24005	.83564	1.19669	7
54	.75082	1.33187	.77848	1.28456	.80690	1.23931	.83613	1.19599	6
55	.75128	1.33107	.77895	1.28379	.80738	1.23858	.83662	1.19528	5
56	.75173	1.33026	.77941	1.28302	.80786	1.23784	.83712	1.19457	4
57	.75219	1.32946	.77988	1.28225	.80834	1.23710	.83761	1.19387	3
58	.75264	1.32865	.78035	1.28148	.80882	1.23637	.83811	1.19316	2
59	.75310	1.32785	.78082	1.28071	.80930	1.23563	.83860	1.19246	1
60	.75355	1.32704	.78129	1.27994	.80978	1.23490	.83910	1.19175	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	53°		52°		51°		50°		

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′	40° Tang	40° Cotang	41° Tang	41° Cotang	42° Tang	42° Cotang	43° Tang	43° Cotang	′
0	.83910	1.19175	.86929	1.15037	.90040	1.11061	.93252	1.07237	60
1	.83960	1.19105	.86980	1.14969	.90093	1.10996	.93306	1.07174	59
2	.84009	1.19035	.87031	1.14902	.90146	1.10931	.93360	1.07112	58
3	.84059	1.18964	.87082	1.14834	.90199	1.10867	.93415	1.07049	57
4	.84108	1.18894	.87133	1.14767	.90251	1.10802	.93469	1.06987	56
5	.84158	1.18824	.87184	1.14699	.90304	1.10737	.93524	1.06925	55
6	.84208	1.18754	.87236	1.14632	.90357	1.10672	.93578	1.06862	54
7	.84258	1.18684	.87287	1.14565	.90410	1.10607	.93633	1.06800	53
8	.84307	1.18614	.87338	1.14498	.90463	1.10543	.93688	1.06738	52
9	.84357	1.18544	.87389	1.14430	.90516	1.10478	.93742	1.06676	51
10	.84407	1.18474	.87441	1.14363	.90569	1.10414	.93797	1.06613	50
11	.84457	1.18404	.87492	1.14296	.90621	1.10349	.93852	1.06551	49
12	.84507	1.18334	.87543	1.14229	.90674	1.10285	.93906	1.06489	48
13	.84556	1.18264	.87595	1.14162	.90727	1.10220	.93961	1.06427	47
14	.84606	1.18194	.87646	1.14095	.90781	1.10156	.94016	1.06365	46
15	.84656	1.18125	.87698	1.14028	.90834	1.10091	.94071	1.06303	45
16	.84706	1.18055	.87749	1.13961	.90887	1.10027	.94125	1.06241	44
17	.84756	1.17986	.87801	1.13894	.90940	1.09963	.94180	1.06179	43
18	.84806	1.17916	.87852	1.13828	.90993	1.09899	.94235	1.06117	42
19	.84856	1.17846	.87904	1.13761	.91046	1.09834	.94290	1.06056	41
20	.84906	1.17777	.87955	1.13694	.91099	1.09770	.94345	1.05994	40
21	.84956	1.17708	.88007	1.13627	.91153	1.09706	.94400	1.05932	39
22	.85006	1.17638	.88059	1.13561	.91206	1.09642	.94455	1.05870	38
23	.85057	1.17569	.88110	1.13494	.91259	1.09578	.94510	1.05809	37
24	.85107	1.17500	.88162	1.13428	.91313	1.09514	.94565	1.05747	36
25	.85157	1.17430	.88214	1.13361	.91366	1.09450	.94620	1.05685	35
26	.85207	1.17361	.88265	1.13295	.91419	1.09386	.94676	1.05624	34
27	.85257	1.17292	.88317	1.13228	.91473	1.09322	.94731	1.05562	33
28	.85308	1.17223	.88369	1.13162	.91526	1.09258	.94786	1.05501	32
29	.85358	1.17154	.88421	1.13096	.91580	1.09195	.94841	1.05439	31
30	.85408	1.17085	.88473	1.13029	.91633	1.09131	.94896	1.05378	30
31	.85458	1.17016	.88524	1.12963	.91687	1.09067	.94952	1.05317	29
32	.85509	1.16947	.88576	1.12897	.91740	1.09003	.95007	1.05255	28
33	.85559	1.16878	.88628	1.12831	.91794	1.08940	.95062	1.05194	27
34	.85609	1.16809	.88680	1.12765	.91847	1.08876	.95118	1.05133	26
35	.85660	1.16741	.88732	1.12699	.91901	1.08813	.95173	1.05072	25
36	.85710	1.16672	.88784	1.12633	.91955	1.08749	.95229	1.05010	24
37	.85761	1.16603	.88836	1.12567	.92008	1.08686	.95284	1.04949	23
38	.85811	1.16535	.88888	1.12501	.92062	1.08622	.95340	1.04888	22
39	.85862	1.16466	.88940	1.12435	.92116	1.08559	.95395	1.04827	21
40	.85912	1.16398	.88992	1.12369	.92170	1.08496	.95451	1.04766	20
41	.85963	1.16329	.89045	1.12303	.92224	1.08432	.95506	1.04705	19
42	.86014	1.16261	.89097	1.12238	.92277	1.08369	.95562	1.04644	18
43	.86064	1.16192	.89149	1.12172	.92331	1.08306	.95618	1.04583	17
44	.86115	1.16124	.89201	1.12106	.92385	1.08243	.95673	1.04522	16
45	.86166	1.16056	.89253	1.12041	.92439	1.08179	.95729	1.04461	15
46	.86216	1.15987	.89306	1.11975	.92493	1.08116	.95785	1.04401	14
47	.86267	1.15919	.89358	1.11909	.92547	1.08053	.95841	1.04340	13
48	.86318	1.15851	.89410	1.11844	.92601	1.07990	.95897	1.04279	12
49	.86368	1.15783	.89463	1.11778	.92655	1.07927	.95952	1.04218	11
50	.86419	1.15715	.89515	1.11713	.92709	1.07864	.96008	1.04158	10
51	.86470	1.15647	.89567	1.11648	.92763	1.07801	.96064	1.04097	9
52	.86521	1.15579	.89620	1.11582	.92817	1.07738	.96120	1.04036	8
53	.86572	1.15511	.89672	1.11517	.92872	1.07676	.96176	1.03976	7
54	.86623	1.15443	.89725	1.11452	.92926	1.07613	.96232	1.03915	6
55	.86674	1.15375	.89777	1.11387	.92980	1.07550	.96288	1.03855	5
56	.86725	1.15308	.89830	1.11321	.93034	1.07487	.96344	1.03794	4
57	.86776	1.15240	.89883	1.11256	.93088	1.07425	.96400	1.03734	3
58	.86827	1.15172	.89935	1.11191	.93143	1.07362	.96457	1.03674	2
59	.86878	1.15104	.89988	1.11126	.93197	1.07299	.96513	1.03613	1
60	.86929	1.15037	.90040	1.11061	.93252	1.07237	.96569	1.03553	0
′	Cotang	Tang	Cotang	Tang	Cotang	Tang	Cotang	Tang	′
	49°		48°		47°		46°		

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′	44° Tang	44° Cotang	′	′	44° Tang	44° Cotang	′	′	44° Tang	44° Cotang	′
0	.96569	1.03553	60	20	.97700	1.02355	40	40	.98843	1.01170	20
1	.96625	1.03493	59	21	.97756	1.02295	39	41	.98901	1.01112	19
2	.96681	1.03433	58	22	.97813	1.02236	38	42	.98958	1.01053	18
3	.96738	1.03372	57	23	.97870	1.02176	37	43	.99016	1.00994	17
4	.96794	1.03312	56	24	.97927	1.02117	36	44	.99073	1.00935	16
5	.96850	1.03252	55	25	.97984	1.02057	35	45	.99131	1.00876	15
6	.96907	1.03192	54	26	.98041	1.01998	34	46	.99189	1.00818	14
7	.96963	1.03132	53	27	.98098	1.01939	33	47	.99247	1.00759	13
8	.97020	1.03072	52	28	.98155	1.01879	32	48	.99304	1.00701	12
9	.97076	1.03012	51	29	.98213	1.01820	31	49	.99362	1.00642	11
10	.97133	1.02952	50	30	.98270	1.01761	30	50	.99420	1.00583	10
11	.97189	1.02892	49	31	.98327	1.01702	29	51	.99478	1.00525	9
12	.97246	1.02832	48	32	.98384	1.01642	28	52	.99536	1.00467	8
13	.97302	1.02772	47	33	.98441	1.01583	27	53	.99594	1.00408	7
14	.97359	1.02713	46	34	.98499	1.01524	26	54	.99652	1.00350	6
15	.97416	1.02653	45	35	.98556	1.01465	25	55	.99710	1.00291	5
16	.97472	1.02593	44	36	.98613	1.01406	24	56	.99768	1.00233	4
17	.97529	1.02533	43	37	.98671	1.01347	23	57	.99826	1.00175	3
18	.97586	1.02474	42	38	.98728	1.01288	22	58	.99884	1.00116	2
19	.97643	1.02414	41	39	.98786	1.01229	21	59	.99942	1.00058	1
20	.97700	1.02355	40	40	.98843	1.01170	20	60	1.00000	1.00000	0
′	Cotang	Tang	′	′	Cotang	Tang	′	′	Cotang	Tang	′
	45°				45°				45°		

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′	0° Sec.	0° Cosec.	1° Sec.	1° Cosec.	2° Sec.	2° Cosec.	3° Sec.	3° Cosec.	′
0	1	Infinite	1.0001	57.299	1.0006	28.654	1.0014	19.107	60
1	1	3437.70	1.0001	56.359	1.0006	28.417	1.0014	19.002	59
2	1	1718.90	1.0002	55.450	1.0006	28.184	1.0014	18.897	58
3	1	1145.90	1.0002	54.570	1.0006	27.955	1.0014	18.794	57
4	1	859.44	1.0002	53.718	1.0006	27.730	1.0014	18.692	56
5	1	687.55	1.0002	52.891	1.0007	27.508	1.0014	18.591	55
6	1	572.96	1.0002	52.090	1.0007	27.290	1.0015	18.491	54
7	1	491.11	1.0002	51.313	1.0007	27.075	1.0015	18.393	53
8	1	429.72	1.0002	50.558	1.0007	26.864	1.0015	18.295	52
9	1	381.97	1.0002	49.826	1.0007	26.655	1.0015	18.198	51
10	1	343.77	1.0002	49.114	1.0007	26.450	1.0015	18.103	50
11	1	312.52	1.0002	48.422	1.0007	26.249	1.0015	18.008	49
12	1	286.48	1.0002	47.750	1.0007	26.050	1.0016	17.914	48
13	1	264.44	1.0002	47.096	1.0007	25.854	1.0016	17.821	47
14	1	245.55	1.0002	46.460	1.0008	25.661	1.0016	17.730	46
15	1	229.18	1.0002	45.840	1.0008	25.471	1.0016	17.639	45
16	1	214.86	1.0002	45.237	1.0008	25.284	1.0016	17.549	44
17	1	202.22	1.0002	44.650	1.0008	25.100	1.0016	17.460	43
18	1	190.99	1.0002	44.077	1.0008	24.918	1.0017	17.372	42
19	1	180.73	1.0003	43.520	1.0008	24.739	1.0017	17.285	41
20	1	171.89	1.0003	42.976	1.0008	24.562	1.0017	17.198	40
21	1	163.70	1.0003	42.445	1.0008	24.388	1.0017	17.113	39
22	1	156.26	1.0003	41.928	1.0008	24.216	1.0017	17.028	38
23	1	149.47	1.0003	41.423	1.0009	24.047	1.0017	16.944	37
24	1	143.24	1.0003	40.930	1.0009	23.880	1.0018	16.861	36
25	1	137.51	1.0003	40.448	1.0009	23.716	1.0018	16.779	35
26	1	132.22	1.0003	39.978	1.0009	23.553	1.0018	16.698	34
27	1	127.32	1.0003	39.518	1.0009	23.393	1.0018	16.617	33
28	1	122.78	1.0003	39.069	1.0009	23.235	1.0018	16.538	32
29	1	118.54	1.0003	38.631	1.0009	23.079	1.0018	16.459	31
30	1	114.59	1.0003	38.201	1.0009	22.925	1.0019	16.380	30
31	1	110.90	1.0003	37.782	1.0010	22.774	1.0019	16.303	29
32	1	107.43	1.0003	37.371	1.0010	22.624	1.0019	16.226	28
33	1	104.17	1.0004	36.969	1.0010	22.476	1.0019	16.150	27
34	1	101.11	1.0004	36.576	1.0010	22.330	1.0019	16.075	26
35	1	98.223	1.0004	36.191	1.0010	22.186	1.0019	16.000	25
36	1	95.495	1.0004	35.814	1.0010	22.044	1.0020	15.926	24
37	1	92.914	1.0004	35.445	1.0010	21.904	1.0020	15.853	23
38	1.0001	90.469	1.0004	35.084	1.0010	21.765	1.0020	15.780	22
39	1.0001	88.149	1.0004	34.729	1.0011	21.629	1.0020	15.708	21
40	1.0001	85.946	1.0004	34.382	1.0011	21.494	1.0020	15.637	20
41	1.0001	83.849	1.0004	34.042	1.0011	21.360	1.0021	15.566	19
42	1.0001	81.853	1.0004	33.708	1.0011	21.228	1.0021	15.496	18
43	1.0001	79.950	1.0004	33.381	1.0011	21.098	1.0021	15.427	17
44	1.0001	78.133	1.0004	33.060	1.0011	20.970	1.0021	15.358	16
45	1.0001	76.396	1.0005	32.745	1.0011	20.843	1.0021	15.290	15
46	1.0001	74.736	1.0005	32.437	1.0012	20.717	1.0022	15.222	14
47	1.0001	73.146	1.0005	32.134	1.0012	20.593	1.0022	15.155	13
48	1.0001	71.622	1.0005	31.836	1.0012	20.471	1.0022	15.089	12
49	1.0001	70.160	1.0005	31.544	1.0012	20.350	1.0022	15.023	11
50	1.0001	68.757	1.0005	31.257	1.0012	20.230	1.0022	14.958	10
51	1.0001	67.409	1.0005	30.976	1.0012	20.112	1.0023	14.893	9
52	1.0001	66.113	1.0005	30.699	1.0012	19.995	1.0023	14.829	8
53	1.0001	64.866	1.0005	30.428	1.0013	19.880	1.0023	14.765	7
54	1.0001	63.664	1.0005	30.161	1.0013	19.766	1.0023	14.702	6
55	1.0001	62.507	1.0005	29.899	1.0013	19.653	1.0023	14.640	5
56	1.0001	61.391	1.0006	29.641	1.0013	19.541	1.0024	14.578	4
57	1.0001	60.314	1.0006	29.388	1.0013	19.431	1.0024	14.517	3
58	1.0001	59.274	1.0006	29.139	1.0013	19.322	1.0024	14.456	2
59	1.0001	58.270	1.0006	28.894	1.0013	19.214	1.0024	14.395	1
60	1.0001	57.299	1.0006	28.654	1.0014	19.107	1.0024	14.335	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	89°		88°		87°		86°		

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′	4° Sec.	4° Cosec.	5° Sec.	5° Cosec.	6° Sec.	6° Cosec.	7° Sec.	7° Cosec.	′
0	1.0024	14.335	1.0038	11.474	1.0055	9.5668	1.0075	8.2055	60
1	1.0025	14.276	1.0038	11.436	1.0055	9.5404	1.0075	8.1861	59
2	1.0025	14.217	1.0039	11.398	1.0056	9.5141	1.0076	8.1668	58
3	1.0025	14.159	1.0039	11.360	1.0056	9.4880	1.0076	8.1476	57
4	1.0025	14.101	1.0039	11.323	1.0056	9.4620	1.0076	8.1285	56
5	1.0025	14.043	1.0039	11.286	1.0057	9.4362	1.0077	8.1094	55
6	1.0026	13.986	1.0040	11.249	1.0057	9.4105	1.0077	8.0905	54
7	1.0026	13.930	1.0040	11.213	1.0057	9.3850	1.0078	8.0717	53
8	1.0026	13.874	1.0040	11.176	1.0057	9.3596	1.0078	8.0529	52
9	1.0026	13.818	1.0040	11.140	1.0058	9.3343	1.0078	8.0342	51
10	1.0026	13.763	1.0041	11.104	1.0058	9.3092	1.0079	8.0156	50
11	1.0027	13.708	1.0041	11.069	1.0058	9.2842	1.0079	7.9971	49
12	1.0027	13.654	1.0041	11.033	1.0059	9.2593	1.0079	7.9787	48
13	1.0027	13.600	1.0041	10.988	1.0059	9.2346	1.0080	7.9604	47
14	1.0027	13.547	1.0042	10.963	1.0059	9.2100	1.0080	7.9421	46
15	1.0027	13.494	1.0042	10.929	1.0060	9.1855	1.0080	7.9240	45
16	1.0028	13.441	1.0042	10.894	1.0060	9.1612	1.0081	7.9059	44
17	1.0028	13.389	1.0043	10.860	1.0060	9.1370	1.0081	7.8879	43
18	1.0028	13.337	1.0043	10.826	1.0061	9.1129	1.0082	7.8700	42
19	1.0028	13.286	1.0043	10.792	1.0061	9.0890	1.0082	7.8522	41
20	1.0029	13.235	1.0043	10.758	1.0061	9.0651	1.0082	7.8344	40
21	1.0029	13.184	1.0044	10.725	1.0062	9.0414	1.0083	7.8168	39
22	1.0029	13.134	1.0044	10.692	1.0062	9.0179	1.0083	7.7992	38
23	1.0029	13.084	1.0044	10.659	1.0062	8.9944	1.0084	7.7817	37
24	1.0029	13.034	1.0044	10.626	1.0063	8.9711	1.0084	7.7642	36
25	1.0030	12.985	1.0045	10.593	1.0063	8.9479	1.0084	7.7469	35
26	1.0030	12.937	1.0045	10.561	1.0063	8.9248	1.0085	7.7296	34
27	1.0030	12.888	1.0045	10.529	1.0064	8.9018	1.0085	7.7124	33
28	1.0030	12.840	1.0046	10.497	1.0064	8.8790	1.0085	7.6953	32
29	1.0031	12.793	1.0046	10.465	1.0064	8.8563	1.0086	7.6783	31
30	1.0031	12.745	1.0046	10.433	1.0065	8.8337	1.0086	7.6613	30
31	1.0031	12.698	1.0046	10.402	1.0065	8.8112	1.0087	7.6444	29
32	1.0031	12.652	1.0047	10.371	1.0065	8.7888	1.0087	7.6276	28
33	1.0032	12.606	1.0047	10.340	1.0066	8.7665	1.0087	7.6108	27
34	1.0032	12.560	1.0047	10.309	1.0066	8.7444	1.0088	7.5942	26
35	1.0032	12.514	1.0048	10.278	1.0066	8.7223	1.0088	7.5776	25
36	1.0032	12.469	1.0048	10.248	1.0067	8.7004	1.0089	7.5611	24
37	1.0032	12.424	1.0048	10.217	1.0067	8.6786	1.0089	7.5446	23
38	1.0033	12.379	1.0048	10.187	1.0067	8.6569	1.0089	7.5282	22
39	1.0033	12.335	1.0049	10.157	1.0068	8.6353	1.0090	7.5119	21
40	1.0033	12.291	1.0049	10.127	1.0068	8.6138	1.0090	7.4957	20
41	1.0033	12.248	1.0049	10.098	1.0058	8.5924	1.0090	7.4795	19
42	1.0034	12.204	1.0050	10.068	1.0069	8.5711	1.0091	7.4634	18
43	1.0034	12.161	1.0050	10.039	1.0069	8.5499	1.0091	7.4474	17
44	1.0034	12.118	1.0050	10.010	1.0069	8.5289	1.0092	7.4315	16
45	1.0034	12.076	1.0050	9.9812	1.0070	8.5079	1.0092	7.4156	15
46	1.0035	12.034	1.0051	9.9525	1.0070	8.4871	1.0092	7.3998	14
47	1.0035	11.992	1.0051	9.9239	1.0070	8.4663	1.0093	7.3840	13
48	1.0035	11.950	1.0051	9.8955	1.0071	8.4457	1.0093	7.3683	12
49	1.0035	11.909	1.0052	9.8672	1.0071	8.4251	1.0094	7.3527	11
50	1.0036	11.868	1.0052	9.8391	1.0071	8.4046	1.0094	7.3372	10
51	1.0036	11.828	1.0052	9.8112	1.0072	8.3843	1.0094	7.3217	9
52	1.0036	11.787	1.0053	9.7834	1.0072	8.3640	1.0095	7.3063	8
53	1.0036	11.747	1.0053	9.7558	1.0073	8.3439	1.0095	7.2909	7
54	1.0037	11.707	1.0053	9.7283	1.0073	8.3238	1.0096	7.2757	6
55	1.0037	11.668	1.0053	9.7010	1.0073	8.3039	1.0096	7.2604	5
56	1.0037	11.628	1.0054	9.6739	1.0074	8.2840	1.0097	7.2453	4
57	1.0037	11.589	1.0054	9.6469	1.0074	8.2642	1.0097	7.2302	3
58	1.0038	11.550	1.0054	9.6200	1.0074	8.2446	1.0097	7.2152	2
59	1.0038	11.512	1.0055	9.5933	1.0075	8.2250	1.0098	7.2002	1
60	1.0038	11.474	1.0055	9.5668	1.0075	8.2055	1.0098	7.1853	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	85°		84°		83°		82°		

TM684-170

′	8° Sec.	8° Cosec.	9° Sec.	9° Cosec.	10° Sec.	10° Cosec.	11° Sec.	11° Cosec.	′
0	1.0098	7.1853	1.0125	6.3924	1.0154	5.7588	1.0187	5.2408	60
1	1.0099	7.1704	1.0125	6.3807	1.0155	5.7493	1.0188	5.2330	59
2	1.0099	7.1557	1.0125	6.3690	1.0155	5.7398	1.0188	5.2252	58
3	1.0099	7.1409	1.0126	6.3574	1.0156	5.7304	1.0189	5.2174	57
4	1.0100	7.1263	1.0126	6.3458	1.0156	5.7210	1.0189	5.2097	56
5	1.0100	7.1117	1.0127	6.3343	1.0157	5.7117	1.0190	5.2019	55
6	1.0101	7.0972	1.0127	6.3228	1.0157	5.7023	1.0191	5.1942	54
7	1.0101	7.0827	1.0128	6.3113	1.0158	5.6930	1.0191	5.1865	53
8	1.0102	7.0683	1.0128	6.2999	1.0158	5.6838	1.0192	5.1788	52
9	1.0102	7.0539	1.0129	6.2885	1.0159	5.6745	1.0192	5.1712	51
10	1.0102	7.0396	1.0129	6.2772	1.0159	5.6653	1.0193	5.1636	50
11	1.0103	7.0254	1.0130	6.2659	1.0160	5.6561	1.0193	5.1560	49
12	1.0103	7.0112	1.0130	6.2546	1.0160	5.6470	1.0194	5.1484	48
13	1.0104	6.9971	1.0131	6.2434	1.0161	5.6379	1.0195	5.1409	47
14	1.0104	6.9830	1.0131	6.2322	1.0162	5.6288	1.0195	5.1333	46
15	1.0104	6.9690	1.0132	6.2211	1.0162	5.6197	1.0196	5.1258	45
16	1.0105	6.9550	1.0132	6.2100	1.0163	5.6107	1.0196	5.1183	44
17	1.0105	6.9411	1.0133	6.1990	1.0163	5.6017	1.0197	5.1109	43
18	1.0106	6.9273	1.0133	6.1880	1.0164	5.5928	1.0198	5.1034	42
19	1.0106	6.9135	1.0134	6.1770	1.0164	5.5838	1.0198	5.0960	41
20	1.0107	6.8998	1.0134	6.1661	1.0165	5.5749	1.0199	5.0886	40
21	1.0107	6.8861	1.0135	6.1552	1.0165	5.5660	1.0199	5.0812	39
22	1.0107	6.8725	1.0135	6.1443	1.0166	5.5572	1.0200	5.0739	38
23	1.0108	6.8589	1.0136	6.1335	1.0166	5.5484	1.0201	5.0666	37
24	1.0108	6.8454	1.0136	6.1227	1.0167	5.5396	1.0201	5.0593	36
25	1.0109	6.8320	1.0136	6.1120	1.0167	5.5308	1.0202	5.0520	35
26	1.0109	6.8185	1.0137	6.1013	1.0168	5.5221	1.0202	5.0447	34
27	1.0110	6.8052	1.0137	6.0906	1.0169	5.5134	1.0203	5.0375	33
28	1.0110	6.7919	1.0138	6.0800	1.0169	5.5047	1.0204	5.0302	32
29	1.0111	6.7787	1.0138	6.0694	1.0170	5.4960	1.0204	5.0230	31
30	1.0111	6.7655	1.0139	6.0588	1.0170	5.4874	1.0205	5.0158	30
31	1.0111	6.7523	1.0139	6.0483	1.0171	5.4788	1.0205	5.0087	29
32	1.0112	6.7392	1.0140	6.0379	1.0171	5.4702	1.0206	5.0015	28
33	1.0112	6.7262	1.0140	6.0274	1.0172	5.4617	1.0207	4.9944	27
34	1.0113	6.7132	1.0141	6.0170	1.0172	5.4532	1.0207	4.9873	26
35	1.0113	6.7003	1.0141	6.0066	1.0173	5.4447	1.0208	4.9802	25
36	1.0114	6.6874	1.0142	5.9963	1.0174	5.4362	1.0208	4.9732	24
37	1.0114	6.6745	1.0142	5.9860	1.0174	5.4278	1.0209	4.9661	23
38	1.0115	6.6617	1.0143	5.9758	1.0175	5.4194	1.0210	4.9591	22
39	1.0115	6.6490	1.0143	5.9655	1.0175	5.4110	1.0210	4.9521	21
40	1.0115	6.6363	1.0144	5.9554	1.0176	5.4026	1.0211	4.9452	20
41	1.0116	6.6237	1.0144	5.9452	1.0176	5.3943	1.0211	4.9382	19
42	1.0116	6.6111	1.0145	5.9351	1.0177	5.3860	1.0212	4.9313	18
43	1.0117	6.5985	1.0145	5.9250	1.0177	5.3777	1.0213	4.9243	17
44	1.0117	6.5860	1.0146	5.9150	1.0178	5.3695	1.0213	4.9175	16
45	1.0118	6.5736	1.0146	5.9049	1.0179	5.3612	1.0214	4.9106	15
46	1.0118	6.5612	1.0147	5.8950	1.0179	5.3530	1.0215	4.9037	14
47	1.0119	6.5488	1.0147	5.8850	1.0180	5.3449	1.0215	4.8969	13
48	1.0119	6.5365	1.0148	5.8751	1.0180	5.3367	1.0216	4.8901	12
49	1.0119	6.5243	1.0148	5.8652	1.0181	5.3286	1.0216	4.8833	11
50	1.0120	6.5121	1.0149	5.8554	1.0181	5.3205	1.0217	4.8765	10
51	1.0120	6.4999	1.0150	5.8456	1.0182	5.3124	1.0218	4.8697	9
52	1.0121	6.4878	1.0150	5.8358	1.0182	5.3044	1.0218	4.8630	8
53	1.0121	6.4757	1.0151	5.8261	1.0183	5.2963	1.0219	4.8563	7
54	1.0122	6.4637	1.0151	5.8163	1.0184	5.2883	1.0220	4.8496	6
55	1.0122	6.4517	1.0152	5.8067	1.0184	5.2803	1.0220	4.8429	5
56	1.0123	6.4398	1.0152	5.7970	1.0185	5.2724	1.0221	4.8362	4
57	1.0123	6.4279	1.0153	5.7874	1.0185	5.2645	1.0221	4.8296	3
58	1.0124	6.4160	1.0153	5.7778	1.0186	5.2566	1.0222	4.8229	2
59	1.0124	6.4042	1.0154	5.7683	1.0186	5.2487	1.0223	4.8163	1
60	1.0125	6.3924	1.0154	5.7588	1.0187	5.2408	1.0223	4.8097	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	81°		80°		79°		78°		

TM684-171

′	12° Sec.	12° Cosec.	13° Sec.	13° Cosec.	14° Sec.	14° Cosec.	15° Sec.	15° Cosec.	′
0	1.0223	4.8097	1.0263	4.4454	1.0306	4.1336	1.0353	3.8637	60
1	1.0224	4.8032	1.0264	4.4398	1.0307	4.1287	1.0353	3.8595	59
2	1.0225	4.7966	1.0264	4.4342	1.0308	4.1239	1.0354	3.8553	58
3	1.0225	4.7901	1.0265	4.4287	1.0308	4.1191	1.0355	3.8512	57
4	1.0226	4.7835	1.0266	4.4231	1.0309	4.1144	1.0356	3.8470	56
5	1.0226	4.7770	1.0266	4.4176	1.0310	4.1096	1.0357	3.8428	55
6	1.0227	4.7706	1.0267	4.4121	1.0311	4.1048	1.0358	3.8387	54
7	1.0228	4.7641	1.0268	4.4065	1.0311	4.1001	1.0358	3.8346	53
8	1.0228	4.7576	1.0268	4.4011	1.0312	4.0953	1.0359	3.8304	52
9	1.0229	4.7512	1.0269	4.3956	1.0313	4.0906	1.0360	3.8263	51
10	1.0230	4.7448	1.0270	4.3901	1.0314	4.0859	1.0361	3.8222	50
11	1.0230	4.7384	1.0271	4.3847	1.0314	4.0812	1.0362	3.8181	49
12	1.0231	4.7320	1.0271	4.3792	1.0315	4.0765	1.0362	3.8140	48
13	1.0232	4.7257	1.0272	4.3738	1.0316	4.0718	1.0363	3.8100	47
14	1.0232	4.7193	1.0273	4.3684	1.0317	4.0672	1.0364	3.8059	46
15	1.0233	4.7130	1.0273	4.3630	1.0317	4.0625	1.0365	3.8018	45
16	1.0234	4.7067	1.0274	4.3576	1.0318	4.0579	1.0366	3.7978	44
17	1.0234	4.7004	1.0275	4.3522	1.0319	4.0532	1.0367	3.7937	43
18	1.0235	4.6942	1.0276	4.3469	1.0320	4.0486	1.0367	3.7897	42
19	1.0235	4.6879	1.0276	4.3415	1.0320	4.0440	1.0368	3.7857	41
20	1.0236	4.6817	1.0277	4.3362	1.0321	4.0394	1.0369	3.7816	40
21	1.0237	4.6754	1.0278	4.3309	1.0322	4.0348	1.0370	3.7776	39
22	1.0237	4.6692	1.0278	4.3256	1.0323	4.0302	1.0371	3.7736	38
23	1.0238	4.6631	1.0279	4.3203	1.0323	4.0256	1.0371	3.7697	37
24	1.0239	4.6569	1.0280	4.3150	1.0324	4.0211	1.0372	3.7657	36
25	1.0239	4.6507	1.0280	4.3098	1.0325	4.0165	1.0373	3.7617	35
26	1.0240	4.6446	1.0281	4.3045	1.0326	4.0120	1.0374	3.7577	34
27	1.0241	4.6385	1.0282	4.2993	1.0327	4.0074	1.0375	3.7538	33
28	1.0241	4.6324	1.0283	4.2941	1.0327	4.0029	1.0376	3.7498	32
29	1.0242	4.6263	1.0283	4.2888	1.0328	3.9984	1.0376	3.7459	31
30	1.0243	4.6202	1.0284	4.2836	1.0329	3.9939	1.0377	3.7420	30
31	1.0243	4.6142	1.0285	4.2785	1.0330	3.9894	1.0378	3.7380	29
32	1.0244	4.6081	1.0285	4.2733	1.0330	3.9850	1.0379	3.7341	28
33	1.0245	4.6021	1.0286	4.2681	1.0331	3.9805	1.0380	3.7302	27
34	1.0245	4.5961	1.0287	4.2630	1.0332	3.9760	1.0381	3.7263	26
35	1.0246	4.5901	1.0288	4.2579	1.0333	3.9716	1.0382	3.7224	25
36	1.0247	4.5841	1.0288	4.2527	1.0334	3.9672	1.0382	3.7186	24
37	1.0247	4.5782	1.0289	4.2476	1.0334	3.9627	1.0383	3.7147	23
38	1.0248	4.5722	1.0290	4.2425	1.0335	3.9583	1.0384	3.7108	22
39	1.0249	4.5663	1.0291	4.2375	1.0336	3.9539	1.0385	3.7070	21
40	1.0249	4.5604	1.0291	4.2324	1.0337	3.9495	1.0386	3.7031	20
41	1.0250	4.5545	1.0292	4.2273	1.0338	3.9451	1.0387	3.6993	19
42	1.0251	4.5486	1.0293	4.2223	1.0338	3.9408	1.0387	3.6955	18
43	1.0251	4.5428	1.0293	4.2173	1.0339	3.9364	1.0388	3.6917	17
44	1.0252	4.5369	1.0294	4.2122	1.0340	3.9320	1.0389	3.6878	16
45	1.0253	4.5311	1.0295	4.2072	1.0341	3.9277	1.0390	3.6840	15
46	1.0253	4.5253	1.0296	4.2022	1.0341	3.9234	1.0391	3.6802	14
47	1.0254	4.5195	1.0296	4.1972	1.0342	3.9190	1.0392	3.6765	13
48	1.0255	4.5137	1.0297	4.1923	1.0343	3.9147	1.0393	3.6727	12
49	1.0255	4.5079	1.0298	4.1873	1.0344	3.9104	1.0393	3.6689	11
50	1.0256	4.5021	1.0299	4.1824	1.0345	3.9061	1.0394	3.6651	10
51	1.0257	4.4964	1.0299	4.1774	1.0345	3.9018	1.0395	3.6614	9
52	1.0257	4.4907	1.0300	4.1725	1.0346	3.8976	1.0396	3.6576	8
53	1.0258	4.4850	1.0301	4.1676	1.0347	3.8933	1.0397	3.6539	7
54	1.0259	4.4793	1.0302	4.1627	1.0348	3.8890	1.0398	3.6502	6
55	1.0260	4.4736	1.0302	4.1578	1.0349	3.8848	1.0399	3.6464	5
56	1.0260	4.4679	1.0303	4.1529	1.0349	3.8805	1.0399	3.6427	4
57	1.0261	4.4623	1.0304	4.1481	1.0350	3.8763	1.0400	3.6390	3
58	1.0262	4.4566	1.0305	4.1432	1.0351	3.8721	1.0401	3.6353	2
59	1.0262	4.4510	1.0305	4.1384	1.0352	3.8679	1.0402	3.6316	1
60	1.0263	4.4454	1.0306	4.1336	1.0353	3.8637	1.0403	3.6279	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	77°		76°		75°		74°		

TM684-172

′	16° Sec.	16° Cosec.	17° Sec.	17° Cosec.	18° Sec.	18° Cosec.	19° Sec.	19° Cosec.	′
0	1.0403	3.6279	1.0457	3.4203	1.0515	3.2361	1.0576	3.0715	60
1	1.0404	3.6243	1.0458	3.4170	1.0516	3.2332	1.0577	[illegible]	59
2	1.0405	3.6206	1.0459	3.4138	1.0517	3.2303	1.0578	3.0664	58
3	1.0406	3.6169	1.0460	3.4106	1.0518	3.2274	1.0579	3.0638	57
4	1.0406	3.6133	1.0461	3.4073	1.0519	3.2245	1.0580	3.0612	56
5	1.0407	3.6096	1.0461	3.4041	1.0520	3.2216	1.0581	3.0586	55
6	1.0408	3.6060	1.0462	3.4009	1.0521	3.2188	1.0582	3.0561	54
7	1.0409	3.6024	1.0463	3.3977	1.0522	3.2159	1.0584	3.0535	53
8	1.0410	3.5987	1.0464	3.3945	1.0523	3.2131	1.0585	3.0509	52
9	1.0411	3.5951	1.0465	3.3913	1.0524	3.2102	1.9586	3.0484	51
10	1.0412	3.5915	1.0466	3.3881	1.0525	3.2074	1.0587	3.0458	50
11	1.0413	3.5879	1.0467	3.3849	1.0526	3.2045	1.0588	3.0433	49
12	1.0413	3.5843	1.0468	3.3817	1.0527	3.2017	1.0589	3.0407	48
13	1.0414	3.5807	1.0469	3.3785	1.0528	3.1989	1.0590	3.0382	47
14	1.0415	3.5772	1.0470	3.3754	1.0529	3.1960	1.0591	3.0357	46
15	1.0416	3.5736	1.0471	3.3722	1.0530	3.1932	1.0592	3.0331	45
16	1.0417	3.5700	1.0472	3.3690	1.0531	3.1904	1.0593	3.0306	44
17	1.0418	3.5665	1.0473	3.3659	1.0532	3.1876	1.0594	3.0281	43
18	1.0419	3.5629	1.0474	3.3627	1.0533	3.1848	1.0595	3.0256	42
19	1.0420	3.5594	1.0475	3.3596	1.0534	3.1820	1.0596	3.0231	41
20	1.0420	3.5559	1.0476	3.3565	1.0535	3.1792	1.0598	3.0206	40
21	1.0421	3.5523	1.0477	3.3534	1.0536	3.1764	1.0599	3.0181	39
22	1.0422	3.5488	1.0478	3.3502	1.0537	3.1736	1.0600	3.0156	38
23	1.0423	3.5453	1.0478	3.3471	1.0538	3.1708	1.0601	3.0131	37
24	1.0424	3.5418	1.0479	3.3440	1.0539	3.1681	1.0602	3.0106	36
25	1.0425	3.5383	1.0480	3.3409	1.0540	3.1653	1.0603	3.0081	35
26	1.0426	3.5348	1.0481	3.3378	1.0541	3.1625	1.0604	3.0056	34
27	1.0427	3.5313	1.0482	3.3347	1.0542	3.1598	1.0605	3.0031	33
28	1.0428	3.5279	1.0483	3.3316	1.0543	3.1570	1.0606	3.0007	32
29	1.0428	3.5244	1.0484	3.3286	1.0544	3.1543	1.0607	2.9982	31
30	1.0429	3.5209	1.0485	3.3255	1.0545	3.1515	1.0608	2.9957	30
31	1.0430	3.5175	1.0486	3.3224	1.0546	3.1488	1.0609	2.9933	29
32	1.0431	3.5140	1.0487	3.3194	1.0547	3.1461	1.0611	2.9908	28
33	1.0432	3.5106	1.0488	3.3163	1.0548	3.1433	1.0612	2.9884	27
34	1.0433	3.5072	1.0489	3.3133	1.0549	3.1406	1.0613	2.9859	26
35	1.0434	3.5037	1.0490	3.3102	1.0550	3.1379	1.0614	2.9835	25
36	1.0435	3.5003	1.0491	3.3072	1.0551	3.1352	1.0615	2.9810	24
37	1.0436	3.4969	1.0492	3.3042	1.0552	3.1325	1.0616	2.9786	23
38	1.0437	3.4935	1.0493	3.3011	1.0553	3.1298	1.0617	2.9762	22
39	1.0438	3.4901	1.0494	3.2981	1.0554	3.1271	1.0618	2.9738	21
40	1.0438	3.4867	1.0495	3.2951	1.0555	3.1244	1.0619	2.9713	20
41	1.0439	3.4833	1.0496	3.2921	1.0556	3.1217	1.0620	2.9689	19
42	1.0440	3.4799	1.0497	3.2891	1.0557	3.1190	1.0622	2.9665	18
43	1.0441	3.4766	1.0498	3.2861	1.0558	3.1163	1.0623	2.9641	17
44	1.0442	3.4732	1.0499	3.2831	1.0559	3.1137	1.0624	2.9617	16
45	1.0443	3.4698	1.0500	3.2801	1.0560	3.1110	1.0625	2.9593	15
46	1.0444	3.4665	1.0501	3.2772	1.0561	3.1083	1.0626	2.9569	14
47	1.0445	3.4632	1.0502	3.2742	1.0562	3.1057	1.0627	2.9545	13
48	1.0446	3.4598	1.0503	3.2712	1.0563	3.1030	1.0628	2.9521	12
49	1.0447	3.4565	1.0504	3.2683	1.0565	3.1004	1.0629	2.9497	11
50	1.0448	3.4532	1.0505	3.2653	1.0566	3.0977	1.0630	2.9474	10
51	1.0448	3.4498	1.0506	3.2624	1.0567	3.0951	1.0632	2.9450	9
52	1.0449	3.4465	1.0507	3.2594	1.0568	3.0925	1.0633	2.9426	8
53	1.0450	3.4432	1.0508	3.2565	1.0569	3.0898	1.0634	2.9402	7
54	1.0451	3.4399	1.0509	3.2535	1.0570	3.0872	1.0635	2.9379	6
55	1.0452	3.4366	1.0510	3.2506	1.0571	3.0846	1.0636	2.9355	5
56	1.0453	3.4334	1.0511	3.2477	1.0572	3.0820	1.0637	2.9332	4
57	1.0454	3.4301	1.0512	3.2448	1.0573	3.0793	1.0638	2.9308	3
58	1.0455	3.4268	1.0513	3.2419	1.0574	3.0767	1.0639	2.9285	2
59	1.0456	3.4236	1.0514	3.2390	1.0575	3.0741	1.0641	2.9261	1
60	1.0457	3.4203	1.0515	3.2361	1.0576	3.0715	1.0642	2.9238	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	73°		72°		71°		70°		

TM684-173

′	20° Sec.	20° Cosec.	21° Sec.	21° Cosec.	22° Sec.	22° Cosec.	23° Sec.	23° Cosec.	′
0	1.0642	2.9238	1.0711	2.7904	1.0785	2.6695	1.0864	2.5593	60
1	1.0643	2.9215	1.0713	2.7883	1.0787	2.6675	1.0865	2.5575	59
2	1.0644	2.9191	1.0714	2.7862	1.0788	2.6656	1.0866	2.5558	58
3	1.0645	2.9168	1.0715	2.7841	1.0789	2.6637	1.0868	2.5540	57
4	1.0646	2.9145	1.0716	2.7820	1.0790	2.6618	1.0869	2.5523	56
5	1.0647	2.9122	1.0717	2.7799	1.0792	2.6599	1.0870	2.5506	55
6	1.0648	2.9098	1.0719	2.7778	1.0793	2.6580	1.0872	2.5488	54
7	1.0650	2.9075	1.0720	2.7757	1.0794	2.6561	1.0873	2.5471	53
8	1.0651	2.9052	1.0721	2.7736	1.0795	2.6542	1.0874	2.5453	52
9	1.0652	2.9029	1.0722	2.7715	1.0797	2.6523	1.0876	2.5436	51
10	1.0653	2.9006	1.0723	2.7694	1.0798	2.6504	1.0877	2.5419	50
11	1.0654	2.8983	1.0725	2.7674	1.0799	2.6485	1.0878	2.5402	49
12	1.0655	2.8960	1.0726	2.7653	1.0801	2.6466	1.0880	2.5384	48
13	1.0656	2.8937	1.0727	2.7632	1.0802	2.6447	1.0881	2.5367	47
14	1.0658	2.8915	1.0728	2.7611	1.0803	2.6428	1.0882	2.5350	46
15	1.0659	2.8892	1.0729	2.7591	1.0804	2.6410	1.0884	2.5333	45
16	1.0660	2.8869	1.0731	2.7570	1.0806	2.6391	1.0885	2.5316	44
17	1.0661	2.8846	1.0732	2.7550	1.0807	2.6372	1.0886	2.5299	43
18	1.0662	2.8824	1.0733	2.7529	1.0808	2.6353	1.0888	2.5281	42
19	1.0663	2.8801	1.0734	2.7509	1.0810	2.6335	1.0889	2.5264	41
20	1.0664	2.8778	1.0736	2.7488	1.0811	2.6316	1.0891	2.5247	40
21	1.0666	2.8756	1.0737	2.7468	1.0812	2.6297	1.0892	2.5230	39
22	1.0667	2.8733	1.0738	2.7447	1.0813	2.6279	1.0893	2.5213	38
23	1.0668	2.8711	1.0739	2.7427	1.0815	2.6260	1.0895	2.5196	37
24	1.0669	2.8688	1.0740	2.7406	1.0816	2.6242	1.0896	2.5179	36
25	1.0670	2.8666	1.0742	2.7386	1.0817	2.6223	1.0897	2.5163	35
26	1.0671	2.8644	1.0743	2.7366	1.0819	2.6205	1.0899	2.5146	34
27	1.0673	2.8621	1.0744	2.7346	1.0820	2.6186	1.0900	2.5129	33
28	1.0674	2.8599	1.0745	2.7325	1.0821	2.6168	1.0902	2.5112	32
29	1.0675	2.8577	1.0747	2.7305	1.0823	2.6150	1.0903	2.5095	31
30	1.0676	2.8554	1.0748	2.7285	1.0824	2.6131	1.0904	2.5078	30
31	1.0677	2.8532	1.0749	2.7265	1.0825	2.6113	1.0906	2.5062	29
32	1.0678	2.8510	1.0750	2.7245	1.0826	2.6095	1.0907	2.5045	28
33	1.0679	2.8488	1.0751	2.7225	1.0828	2.6076	1.0908	2.5028	27
34	1.0681	2.8466	1.0753	2.7205	1.0829	2.6058	1.0910	2.5011	26
35	1.0682	2.8444	1.0754	2.7185	1.0830	2.6040	1.0911	2.4995	25
36	1.0683	2.8422	1.0755	2.7165	1.0832	2.6022	1.0913	2.4978	24
37	1.0684	2.8400	1.0756	2.7145	1.0833	2.6003	1.0914	2.4961	23
38	1.0685	2.8378	1.0758	2.7125	1.0834	2.5985	1.0915	2.4945	22
39	1.0686	2.8356	1.0759	2.7105	1.0836	2.5967	1.0917	2.4928	21
40	1.0688	2.8334	1.0760	2.7085	1.0837	2.5949	1.0918	2.4912	20
41	1.0689	2.8312	1.0761	2.7065	1.0838	2.5931	1.0920	2.4895	19
42	1.0690	2.8290	1.0763	2.7045	1.0840	2.5913	1.0921	2.4879	18
43	1.0691	2.8269	1.0764	2.7026	1.0841	2.5895	1.0922	2.4862	17
44	1.0692	2.8247	1.0765	2.7006	1.0842	2.5877	1.0924	2.4846	16
45	1.0694	2.8225	1.0766	2.6986	1.0844	2.5859	1.0925	2.4829	15
46	1.0695	2.8204	1.0768	2.6967	1.0845	2.5841	1.0927	2.4813	14
47	1.0696	2.8182	1.0769	2.6947	1.0846	2.5823	1.0928	2.4797	13
48	1.0697	2.8160	1.0770	2.6927	1.0847	2.5805	1.0929	2.4780	12
49	1.0698	2.8139	1.0771	2.6908	1.0849	2.5787	1.0931	2.4764	11
50	1.0699	2.8117	1.0773	2.6888	1.0850	2.5770	1.0932	2.4748	10
51	1.0701	2.8096	1.0774	2.6869	1.0851	2.5752	1.0934	2.4731	9
52	1.0702	2.8074	1.0775	2.6849	1.0853	2.5734	1.0935	2.4715	8
53	1.0703	2.8053	1.0776	2.6830	1.0854	2.5716	1.0936	2.4699	7
54	1.0704	2.8032	1.0778	2.6810	1.0855	2.5699	1.0938	2.4683	6
55	1.0705	2.8010	1.0779	2.6791	1.0857	2.5681	1.0939	2.4666	5
56	1.0707	2.7989	1.0780	2.6772	1.0858	2.5663	1.0941	2.4650	4
57	1.0708	2.7968	1.0781	2.6752	1.0859	2.5646	1.0942	2.4634	3
58	1.0709	2.7947	1.0783	2.6733	1.0861	2.5628	1.0943	2.4618	2
59	1.0710	2.7925	1.0784	2.6714	1.0862	2.5610	1.0945	2.4602	1
60	1.0711	2.7904	1.0785	2.6695	1.0864	2.5593	1.0946	2.4586	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	69°		68°		67°		66°		

TM684-174

′	24° Sec.	24° Cosec.	25° Sec.	25° Cosec.	26° Sec.	26° Cosec.	27° Sec.	27° Cosec.	′
0	1.0946	2.4586	1.1034	2.3662	1.1126	2.2812	1.1223	2.2027	60
1	1.0948	2.4570	1.1035	2.3647	1.1127	2.2798	1.1225	2.2014	59
2	1.0949	2.4554	1.1037	2.3632	1.1129	2.2784	1.1226	2.2002	58
3	1.0951	2.4538	1.1038	2.3618	1.1131	2.2771	1.1228	2.1989	57
4	1.0952	2.4522	1.1040	2.3603	1.1132	2.2757	1.1230	2.1977	56
5	1.0953	2.4506	1.1041	2.3588	1.1134	2.2744	1.1231	2.1964	55
6	1.0955	2.4490	1.1043	2.3574	1.1135	2.2730	1.1233	2.1952	54
7	1.0956	2.4474	1.1044	2.3559	1.1137	2.2717	1.1235	2.1939	53
8	1.0958	2.4458	1.1046	2.3544	1.1139	2.2703	1.1237	2.1927	52
9	1.0959	2.4442	1.1047	2.3530	1.1140	2.2690	1.1238	2.1914	51
10	1.0961	2.4426	1.1049	2.3515	1.1142	2.2676	1.1240	2.1902	50
11	1.0962	2.4411	1.1050	2.3501	1.1143	2.2663	1.1242	2.1889	49
12	1.0963	2.4395	1.1052	2.3486	1.1145	2.2650	1.1243	2.1877	48
13	1.0965	2.4379	1.1053	2.3472	1.1147	2.2636	1.1245	2.1865	47
14	1.0966	2.4363	1.1055	2.3457	1.1148	2.2623	1.1247	2.1852	46
15	1.0968	2.4347	1.1056	2.3443	1.1150	2.2610	1.1248	2.1840	45
16	1.0969	2.4332	1.1058	2.3428	1.1151	2.2596	1.1250	2.1828	44
17	1.0971	2.4316	1.1059	2.3414	1.1153	2.2583	1.1252	2.1815	43
18	1.0972	2.4300	1.1061	2.3399	1.1155	2.2570	1.1253	2.1803	42
19	1.0973	2.4285	1.1062	2.3385	1.1156	2.2556	1.1255	2.1791	41
20	1.0975	2.4269	1.1064	2.3371	1.1158	2.2543	1.1257	2.1778	40
21	1.0976	2.4254	1.1065	2.3356	1.1159	2.2530	1.1258	2.1766	39
22	1.0978	2.4238	1.1067	2.3342	1.1161	2.2517	1.1260	2.1754	38
23	1.0979	2.4222	1.1068	2.3328	1.1163	2.2503	1.1262	2.1742	37
24	1.0981	2.4207	1.1070	2.3313	1.1164	2.2490	1.1264	2.1730	36
25	1.0982	2.4191	1.1072	2.3299	1.1166	2.2477	1.1265	2.1717	35
26	1.0984	2.4176	1.1073	2.3285	1.1167	2.2464	1.1267	2.1705	34
27	1.0985	2.4160	1.1075	2.3271	1.1169	2.2451	1.1269	2.1693	33
28	1.0986	2.4145	1.1076	2.3256	1.1171	2.2438	1.1270	2.1681	32
29	1.0988	2.4130	1.1078	2.3242	1.1172	2.2425	1.1272	2.1669	31
30	1.0989	2.4114	1.1079	2.3228	1.1174	2.2411	1.1274	2.1657	30
31	1.0901	2.4099	1.1081	2.3214	1.1176	2.2398	1.1275	2.1645	29
32	1.0992	2.4083	1.1082	2.3200	1.1177	2.2385	1.1277	2.1633	28
33	1.0994	2.4068	1.1084	2.3186	1.1179	2.2372	1.1279	2.1620	27
34	1.0995	2.4053	1.1085	2.3172	1.1180	2.2359	1.1281	2.1608	26
35	1.0997	2.4037	1.1087	2.3158	1.1182	2.2346	1.1282	2.1596	25
36	1.0998	2.4022	1.1088	2.3143	1.1184	2.2333	1.1284	2.1584	24
37	1.1000	2.4007	1.1090	2.3129	1.1185	2.2320	1.1286	2.1572	23
38	1.1001	2.3992	1.1092	2.3115	1.1187	2.2307	1.1287	2.1560	22
39	1.1003	2.3976	1.1093	2.3101	1.1189	2.2294	1.1289	2.1548	21
40	1.1004	2.3961	1.1095	2.3087	1.1190	2.2282	1.1291	2.1536	20
41	1.1005	2.3946	1.1096	2.3073	1.1192	2.2269	1.1293	2.1525	19
42	1.1007	2.3931	1.1098	2.3059	1.1193	2.2256	1.1294	2.1513	18
43	1.1008	2.3916	1.1099	2.3046	1.1195	2.2243	1.1296	2.1501	17
44	1.1010	2.3901	1.1101	2.3032	1.1197	2.2230	1.1298	2.1489	16
45	1.1011	2.3886	1.1102	2.3018	1.1198	2.2217	1.1299	2.1477	15
46	1.1013	2.3871	1.1104	2.3004	1.1200	2.2204	1.1301	2.1465	14
47	1.1014	2.3856	1.1106	2.2990	1.1202	2.2192	1.1303	2.1453	13
48	1.1016	2.3841	1.1107	2.2976	1.1203	2.2179	1.1305	2.1441	12
49	1.1017	2.3826	1.1109	2.2962	1.1205	2.2166	1.1306	2.1430	11
50	1.1019	2.3811	1.1110	2.2949	1.1207	2.2153	1.1308	2.1418	10
51	1.1020	2.3796	1.1112	2.2935	1.1208	2.2141	1.1310	2.1406	9
52	1.1022	2.3781	1.1113	2.2921	1.1210	2.2128	1.1312	2.1394	8
53	1.1023	2.3766	1.1115	2.2907	1.1212	2.2115	1.1313	2.1382	7
54	1.1025	2.3751	1.1116	2.2894	1.1213	2.2103	1.1315	2.1371	6
55	1.1026	2.3736	1.1118	2.2880	1.1215	2.2090	1.1317	2.1359	5
56	1.1028	2.3721	1.1120	2.2866	1.1217	2.2077	1.1319	2.1347	4
57	1.1029	2.3706	1.1121	2.2853	1.1218	2.2065	1.1320	2.1335	3
58	1.1031	2.3691	1.1123	2.2839	1.1220	2.2052	1.1322	2.1324	2
59	1.1032	2.3677	1.1124	2.2825	1.1222	2.2039	1.1324	2.1312	1
60	1.1034	2.3662	1.1126	2.2812	1.1223	2.2027	1.1326	2.1300	0
	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	65°		64°		63°		62°		

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′	28° Sec.	28° Cosec.	29° Sec.	29° Cosec.	30° Sec.	30° Cosec.	31° Sec.	31° Cosec.	′
0	1.1326	2.1300	1.1433	2.0627	1.1547	2.0000	1.1666	1.9416	60
1	1.1327	2.1289	1.1435	2.0616	1.1549	1.9990	1.1668	1.9407	59
2	1.1329	2.1277	1.1437	2.0605	1.1551	1.9980	1.1670	1.9397	58
3	1.1331	2.1266	1.1439	2.0594	1.1553	1.9970	1.1672	1.9388	57
4	1.1333	2.1254	1.1441	2.0583	1.1555	1.9960	1.1674	1.9378	56
5	1.1334	2.1242	1.1443	2.0573	1.1557	1.9950	1.1676	1.9369	55
6	1.1336	2.1231	1.1445	2.0562	1.1559	1.9940	1.1678	1.9360	54
7	1.1338	2.1219	1.1446	2.0551	1.1561	1.9930	1.1681	1.9350	53
8	1.1340	2.1208	1.1448	2.0540	1.1562	1.9920	1.1683	1.9341	52
9	1.1341	2.1196	1.1450	2.0530	1.1564	1.9910	1.1685	1.9332	51
10	1.1343	2.1185	1.1452	2.0519	1.1566	1.9900	1.1687	1.9322	50
11	1.1345	2.1173	1.1454	2.0508	1.1568	1.9890	1.1689	1.9313	49
12	1.1347	2.1162	1.1456	2.0498	1.1570	1.9880	1.1691	1.9304	48
13	1.1349	2.1150	1.1458	2.0487	1.1572	1.9870	1.1693	1.9295	47
14	1.1350	2.1139	1.1459	2.0476	1.1574	1.9860	1.1695	1.9285	46
15	1.1352	2.1127	1.1461	2.0466	1.1576	1.9850	1.1697	1.9276	45
16	1.1354	2.1116	1.1463	2.0455	1.1578	1.9840	1.1699	1.9267	44
17	1.1356	2.1104	1.1465	2.0444	1.1580	1.9830	1.1701	1.9258	43
18	1.1357	2.1093	1.1467	2.0434	1.1582	1.9820	1.1703	1.9248	42
19	1.1359	2.1082	1.1469	2.0423	1.1584	1.9811	1.1705	1.9239	41
20	1.1361	2.1070	1.1471	2.0413	1.1586	1.9801	1.1707	1.9230	40
21	1.1363	2.1059	1.1473	2.0402	1.1588	1.9791	1.1709	1.9221	39
22	1.1365	2.1048	1.1474	2.0392	1.1590	1.9781	1.1712	1.9212	38
23	1.1366	2.1036	1.1476	2.0381	1.1592	1.9771	1.1714	1.9203	37
24	1.1368	2.1025	1.1478	2.0370	1.1594	1.9761	1.1716	1.9193	36
25	1.1370	2.1014	1.1480	2.0360	1.1596	1.9752	1.1718	1.9184	35
26	1.1372	2.1002	1.1482	2.0349	1.1598	1.9742	1.1720	1.9175	34
27	1.1373	2.0991	1.1484	2.0339	1.1600	1.9732	1.1722	1.9166	33
28	1.1375	2.0980	1.1486	2.0329	1.1602	1.9722	1.1724	1.9157	32
29	1.1377	2.0969	1.1488	2.0318	1.1604	1.9713	1.1726	1.9148	31
30	1.1379	2.0957	1.1489	2.0308	1.1606	1.9703	1.1728	1.9139	30
31	1.1381	2.0946	1.1491	2.0297	1.1608	1.9693	1.1730	1.9130	29
32	1.1382	2.0935	1.1493	2.0287	1.1610	1.9683	1.1732	1.9121	28
33	1.1384	2.0924	1.1495	2.0276	1.1612	1.9774	1.1734	1.9112	27
34	1.1386	2.0912	1.1497	2.0266	1.1614	1.9664	1.1737	1.9102	26
35	1.1388	2.0901	1.1499	2.0256	1.1616	1.9654	1.1739	1.9093	25
36	1.1390	2.0890	1.1501	2.0245	1.1618	1.9645	1.1741	1.9084	24
37	1.1391	2.0879	1.1503	2.0235	1.1620	1.9635	1.1743	1.9075	23
38	1.1393	2.0868	1.1505	2.0224	1.1622	1.9625	1.1745	1.9066	22
39	1.1395	2.0857	1.1507	2.0214	1.1624	1.9616	1.1747	1.9057	21
40	1.1397	2.0846	1.1508	2.0204	1.1626	1.9606	1.1749	1.9048	20
41	1.1399	2.0835	1.1510	2.0194	1.1628	1.9596	1.1751	1.9039	19
42	1.1401	2.0824	1.1512	2.0183	1.1630	1.9587	1.1753	1.9030	18
43	1.1402	2.0812	1.1514	2.0173	1.1632	1.9577	1.1756	1.9021	17
44	1.1404	2.0801	1.1516	2.0163	1.1634	1.9568	1.1758	1.9013	16
45	1.1406	2.0790	1.1518	2.0152	1.1636	1.9558	1.1760	1.9084	15
46	1.1408	2.0779	1.1520	2.0142	1.1638	1.9549	1.1762	1.8995	14
47	1.1410	2.0768	1.1522	2.0132	1.1640	1.9539	1.1764	1.8986	13
48	1.1411	2.0757	1.1524	2.0122	1.1642	1.9530	1.1766	1.8977	12
49	1.1413	2.0746	1.1526	2.0111	1.1644	1.9520	1.1768	1.8968	11
50	1.1415	2.0735	1.1528	2.0101	1.1646	1.9510	1.1770	1.8959	10
51	1.1417	2.0725	1.1530	2.0091	1.1648	1.9501	1.1772	1.8950	9
52	1.1419	2.0714	1.1531	2.0081	1.1650	1.9491	1.1775	1.8941	8
53	1.1421	2.0703	1.1533	2.0071	1.1652	1.9482	1.1777	1.8932	7
54	1.1422	2.0692	1.1535	2.0061	1.1654	1.9473	1.1779	1.8924	6
55	1.1424	2.0681	1.1537	2.0050	1.1656	1.9463	1.1781	1.8915	5
56	1.1426	2.0670	1.1539	2.0040	1.1658	1.9454	1.1783	1.8986	4
57	1.1428	2.0659	1.1541	2.0030	1.1660	1.9444	1.1785	1.8897	3
58	1.1430	2.0648	1.1543	2.0020	1.1662	1.9435	1.1787	1.8888	2
59	1.1432	2.0637	1.1545	2.0010	1.1664	1.9425	1.1790	1.8879	1
60	1.1433	2.0627	1.1547	2.0000	1.1666	1.9416	1.1792	1.8871	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	61°		60°		59°		58°		

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′	32° Sec.	32° Cosec.	33° Sec.	33° Cosec.	34° Sec.	34° Cosec.	35° Sec.	35° Cosec.	′
0	1.1792	1.8871	1.1924	1.8361	1.2062	1.7883	1.2208	1.7434	60
1	1.1794	1.8862	1.1926	1.8352	1.2064	1.7875	1.2210	1.7427	59
2	1.1796	1.8853	1.1928	1.8344	1.2067	1.7867	1.2213	1.7420	58
3	1.1798	1.8844	1.1930	1.8336	1.2069	1.7860	1.2215	1.7413	57
4	1.1800	1.8836	1.1933	1.8328	1.2072	1.7852	1.2218	1.7405	56
5	1.1802	1.8827	1.1935	1.8320	1.2074	1.7844	1.2220	1.7398	55
6	1.1805	1.8818	1.1937	1.8311	1.2076	1.7837	1.2223	1.7391	54
7	1.1807	1.8809	1.1939	1.8303	1.2079	1.7829	1.2225	1.7384	53
8	1.1809	1.8801	1.1942	1.8295	1.2081	1.7821	1.2228	1.7377	52
9	1.1811	1.8792	1.1944	1.8287	1.2083	1.7814	1.2230	1.7369	51
10	1.1813	1.8783	1.1946	1.8279	1.2086	1.7806	1.2233	1.7362	50
11	1.1815	1.8775	1.1948	1.8271	1.2088	1.7798	1.2235	1.7355	49
12	1.1818	1.8766	1.1951	1.8263	1.2091	1.7791	1.2238	1.7348	48
13	1.1820	1.8757	1.1953	1.8255	1.2093	1.7783	1.2240	1.7341	47
14	1.1822	1.8749	1.1955	1.8246	1.2095	1.7776	1.2243	1.7334	46
15	1.1824	1.8740	1.1958	1.8238	1.2098	1.7768	1.2245	1.7327	45
16	1.1826	1.8731	1.1960	1.8230	1.2100	1.7760	1.2248	1.7319	44
17	1.1828	1.8723	1.1962	1.8222	1.2103	1.7753	1.2250	1.7312	43
18	1.1831	1.8714	1.1964	1.8214	1.2105	1.7745	1.2253	1.7305	42
19	1.1833	1.8706	1.1967	1.8206	1.2107	1.7738	1.2255	1.7298	41
20	1.1835	1.8697	1.1969	1.8198	1.2110	1.7730	1.2258	1.7291	40
21	1.1837	1.8688	1.1971	1.8190	1.2112	1.7723	1.2260	1.7284	39
22	1.1839	1.8680	1.1974	1.8182	1.2115	1.7715	1.2263	1.7277	38
23	1.1841	1.8671	1.1976	1.8174	1.2117	1.7708	1.2265	1.7270	37
24	1.1844	1.8663	1.1978	1.8166	1.2119	1.7700	1.2268	1.7263	36
25	1.1846	1.8654	1.1980	1.8158	1.2122	1.7693	1.2270	1.7256	35
26	1.1848	1.8646	1.1983	1.8150	1.2124	1.7685	1.2273	1.7249	34
27	1.1850	1.8637	1.1985	1.8142	1.2127	1.7678	1.2276	1.7242	33
28	1.1852	1.8629	1.1987	1.8134	1.2129	1.7670	1.2278	1.7234	32
29	1.1855	1.8620	1.1990	1.8126	1.2132	1.7663	1.2281	1.7227	31
30	1.1857	1.8611	1.1992	1.8118	1.2134	1.7655	1.2283	1.7220	30
31	1.1859	1.8603	1.1994	1.8110	1.2136	1.7648	1.2286	1.7213	29
32	1.1861	1.8595	1.1997	1.8102	1.2139	1.7640	1.2288	1.7206	28
33	1.1863	1.8586	1.1999	1.8094	1.2141	1.7633	1.2291	1.7199	27
34	1.1866	1.8578	1.2001	1.8086	1.2144	1.7625	1.2293	1.7192	26
35	1.1868	1.8569	1.2004	1.8078	1.2146	1.7618	1.2296	1.7185	25
36	1.1870	1.8561	1.2006	1.8070	1.2149	1.7610	1.2298	1.7178	24
37	1.1872	1.8552	1.2008	1.8062	1.2151	1.7603	1.2301	1.7171	23
38	1.1874	1.8544	1.2010	1.8054	1.2153	1.7596	1.2304	1.7164	22
39	1.1877	1.8535	1.2013	1.8047	1.2156	1.7588	1.2306	1.7157	21
40	1.1879	1.8527	1.2015	1.8039	1.2158	1.7581	1.2309	1.7151	20
41	1.1881	1.8519	1.2017	1.8031	1.2161	1.7573	1.2311	1.7144	19
42	1.1883	1.8510	1.2020	1.8023	1.2163	1.7566	1.2314	1.7137	18
43	1.1886	1.8502	1.2022	1.8015	1.2166	1.7559	1.2316	1.7130	17
44	1.1888	1.8493	1.2024	1.8007	1.2168	1.7551	1.2319	1.7123	16
45	1.1890	1.8485	1.2027	1.7999	1.2171	1.7544	1.2322	1.7116	15
46	1.1892	1.8477	1.2029	1.7992	1.2173	1.7537	1.2324	1.7109	14
47	1.1894	1.8468	1.2031	1.7984	1.2175	1.7529	1.2327	1.7102	13
48	1.1897	1.8460	1.2034	1.7976	1.2178	1.7522	1.2329	1.7095	12
49	1.1899	1.8452	1.2036	1.7968	1.2180	1.7514	1.2332	1.7088	11
50	1.1901	1.8443	1.2039	1.7960	1.2183	1.7507	1.2335	1.7081	10
51	1.1903	1.8435	1.2041	1.7953	1.2185	1.7500	1.2337	1.7075	9
52	1.1906	1.8427	1.2043	1.7945	1.2188	1.7493	1.2340	1.7068	8
53	1.1908	1.8418	1.2046	1.7937	1.2190	1.7485	1.2342	1.7061	7
54	1.1910	1.8410	1.2048	1.7929	1.2193	1.7478	1.2345	1.7054	6
55	1.1912	1.8402	1.2050	1.7921	1.2195	1.7471	1.2348	1.7047	5
56	1.1915	1.8394	1.2053	1.7914	1.2198	1.7463	1.2350	1.7040	4
57	1.1917	1.8385	1.2055	1.7906	1.2200	1.7456	1.2353	1.7033	3
58	1.1919	1.8377	1.2057	1.7898	1.2203	1.7449	1.2355	1.7027	2
59	1.1921	1.8369	1.2060	1.7891	1.2205	1.7442	1.2358	1.7020	1
60	1.1924	1.8361	1.2062	1.7883	1.2208	1.7434	1.2361	1.7013	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	57°		56°		55°		54°		

′	36° Sec.	36° Cosec.	37° Sec.	37° Cosec.	38° Sec.	38° Cosec.	39° Sec.	39° Cosec.	′
0	1.2361	1.7013	1.2521	1.6616	1.2690	1.6243	1.2867	1.5890	60
1	1.2363	1.7006	1.2524	1.6610	1.2693	1.6237	1.2871	1.5884	59
2	1.2366	1.6999	1.2527	1.6603	1.2696	1.6231	1.2874	1.5879	58
3	1.2368	1.6993	1.2530	1.6597	1.2699	1.6224	1.2877	1.5873	57
4	1.2371	1.6986	1.2532	1.6591	1.2702	1.6218	1.2880	1.5867	56
5	1.2374	1.6979	1.2535	1.6584	1.2705	1.6212	1.2883	1.5862	55
6	1.2376	1.6972	1.2538	1.6578	1.2707	1.6206	1.2886	1.5856	54
7	1.2379	1.6965	1.2541	1.6572	1.2710	1.6200	1.2889	1.5850	53
8	1.2382	1.6959	1.2543	1.6565	1.2713	1.6194	1.2892	1.5845	52
9	1.2384	1.6952	1.2546	1.6559	1.2716	1.6188	1.2895	1.5839	51
10	1.2387	1.6945	1.2549	1.6552	1.2719	1.6182	1.2898	1.5833	50
11	1.2389	1.6938	1.2552	1.6546	1.2722	1.6176	1.2901	1.5828	49
12	1.2392	1.6932	1.2554	1.6540	1.2725	1.6170	1.2904	1.5822	48
13	1.2395	1.6925	1.2557	1.6533	1.2728	1.6164	1.2997	1.5816	47
14	1.2397	1.6918	1.2560	1.6527	1.2731	1.6159	1.2910	1.5811	46
15	1.2400	1.6912	1.2563	1.6521	1.2734	1.6153	1.2913	1.5805	45
16	1.2403	1.6905	1.2565	1.6514	1.2737	1.6147	1.2916	1.5799	44
17	1.2405	1.6898	1.2568	1.6508	1.2739	1.6141	1.2919	1.5794	43
18	1.2408	1.6891	1.2571	1.6502	1.2742	1.6135	1.2922	1.5788	42
19	1.2411	1.6885	1.2574	1.6496	1.2745	1.6129	1.2925	1.5783	41
20	1.2413	1.6878	1.2577	1.6489	1.2748	1.6123	1.2929	1.5777	40
21	1.2416	1.6871	1.2579	1.6483	1.2751	1.6117	1.2932	1.5771	39
22	1.2419	1.6865	1.2582	1.6477	1.2754	1.6111	1.2935	1.5766	38
23	1.2421	1.6858	1.2585	1.6470	1.2757	1.6105	1.2938	1.5760	37
24	1.2424	1.6851	1.2588	1.6464	1.2760	1.6099	1.2941	1.5755	36
25	1.2427	1.6845	1.2591	1.6458	1.2763	1.6093	1.2944	1.5749	35
26	1.2429	1.6838	1.2593	1.6452	1.2766	1.6087	1.2947	1.5743	34
27	1.2432	1.6831	1.2596	1.6445	1.2769	1.6081	1.2950	1.5738	33
28	1.2435	1.6825	1.2599	1.6439	1.2772	1.6077	1.2953	1.5732	32
29	1.2437	1.6818	1.2602	1.6433	1.2775	1.6070	1.2956	1.5727	31
30	1.2440	1.6812	1.2605	1.6427	1.2778	1.6064	1.2960	1.5721	30
31	1.2443	1.6805	1.2607	1.6420	1.2781	1.6058	1.2963	1.5716	29
32	1.2445	1.6798	1.2610	1.6414	1.2784	1.6052	1.2966	1.5710	28
33	1.2448	1.6792	1.2613	1.6408	1.2787	1.6046	1.2969	1.5705	27
34	1.2451	1.6785	1.2616	1.6402	1.2790	1.6040	1.2972	1.5699	26
35	1.2453	1.6779	1.2619	1.6396	1.2793	1.6034	1.2975	1.5694	25
36	1.2456	1.6772	1.2622	1.6389	1.2795	1.6029	1.2978	1.5688	24
37	1.2459	1.6766	1.2624	1.6383	1.2798	1.6023	1.2981	1.5683	23
38	1.2461	1.6759	1.2627	1.6377	1.2801	1.6017	1.2985	1.5677	22
39	1.2464	1.6752	1.2630	1.6371	1.2804	1.6011	1.2988	1.5672	21
40	1.2467	1.6746	1.2633	1.6365	1.2807	1.6005	1.2991	1.5666	20
41	1.2470	1.6739	1.2636	1.6359	1.2810	1.6000	1.2994	1.5661	19
42	1.2472	1.6733	1.2639	1.6352	1.2813	1.5994	1.2997	1.5655	18
43	1.2475	1.6726	1.2641	1.6346	1.2816	1.5988	1.3000	1.5650	17
44	1.2478	1.6720	1.2644	1.6340	1.2819	1.5982	1.3003	1.5644	16
45	1.2480	1.6713	1.2647	1.6334	1.2822	1.5976	1.3006	1.5639	15
46	1.2483	1.6707	1.2650	1.6328	1.2825	1.5971	1.3010	1.5633	14
47	1.2486	1.6700	1.2653	1.6322	1.2828	1.5965	1.3013	1.5628	13
48	1.2488	1.6694	1.2656	1.6316	1.2831	1.5959	1.3016	1.5622	12
49	1.2490	1.6687	1.2659	1.6309	1.2834	1.5953	1.3019	1.5617	11
50	1.2494	1.6681	1.2661	1.6303	1.2837	1.5947	1.3022	1.5611	10
51	1.2497	1.6674	1.2664	1.6297	1.2840	1.5942	1.3025	1.5606	9
52	1.2499	1.6668	1.2667	1.6291	1.2843	1.5936	1.3029	1.5600	8
53	1.2502	1.6661	1.2670	1.6285	1.2846	1.5930	1.3032	1.5595	7
54	1.2505	1.6655	1.2673	1.6279	1.2849	1.5924	1.3035	1.5590	6
55	1.2508	1.6648	1.2676	1.6273	1.2852	1.5919	1.3038	1.5584	5
56	1.2510	1.6642	1.2679	1.6267	1.2855	1.5913	1.3041	1.5579	4
57	1.2513	1.6636	1.2681	1.6261	1.2858	1.5907	1.3044	1.5573	3
58	1.2516	1.6629	1.2684	1.6255	1.2861	1.5901	1.3048	1.5568	2
59	1.2519	1.6623	1.2687	1.6249	1.2864	1.5896	1.3051	1.5563	1
60	1.2521	1.6616	1.2690	1.6243	1.2867	1.5890	1.3054	1.5557	0
′	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	Cosec.	Sec.	′
	53°		52°		51°		50°		

′	40° Sec.	40° Cosec.	41° Sec.	41° Cosec.	42° Sec.	42° Cosec.	43° Sec.	43° Cosec.	′
0	1.3054	1.5557	1.3250	1.5242	1.3456	1.4945	1.3673	1.4663	60
1	1.3057	1.5552	1.3253	1.5237	1.3460	1.4940	1.3677	1.4658	59
2	1.3060	1.5546	1.3257	1.5232	1.3463	1.4935	1.3681	1.4654	58
3	1.3064	1.5541	1.3260	1.5227	1.3467	1.4930	1.3684	1.4649	57
4	1.3067	1.5536	1.3263	1.5222	1.3470	1.4925	1.3688	1.4644	56
5	1.3070	1.5530	1.3267	1.5217	1.3474	1.4921	1.3692	1.4640	55
6	1.3073	1.5525	1.3270	1.5212	1.3477	1.4916	1.3695	1.4635	54
7	1.3076	1.5520	1.3274	1.5207	1.3481	1.4911	1.3699	1.4631	53
8	1.3080	1.5514	1.3277	1.5202	1.3485	1.4906	1.3703	1.4626	52
9	1.3083	1.5509	1.3280	1.5197	1.3488	1.4901	1.3707	1.4622	51
10	1.3086	1.5503	1.3284	1.5192	1.3492	1.4897	1.3710	1.4617	50
11	1.3089	1.5498	1.3287	1.5187	1.3495	1.4892	1.3714	1.4613	49
12	1.3092	1.5493	1.3290	1.5182	1.3499	1.4887	1.3718	1.4608	48
13	1.3096	1.5487	1.3294	1.5177	1.3502	1.4882	1.3722	1.4604	47
14	1.3099	1.5482	1.3297	1.5171	1.3506	1.4877	1.3725	1.4599	46
15	1.3102	1.5477	1.3301	1.5166	1.3509	1.4873	1.3729	1.4595	45
16	1.3105	1.5471	1.3304	1.5161	1.3513	1.4868	1.3733	1.4590	44
17	1.3109	1.5466	1.3307	1.5156	1.3517	1.4863	1.3737	1.4586	43
18	1.3112	1.5461	1.3311	1.5151	1.3520	1.4858	1.3740	1.4581	42
19	1.3115	1.5456	1.3314	1.5146	1.3524	1.4854	1.3744	1.4577	41
20	1.3118	1.5450	1.3318	1.5141	1.3527	1.4849	1.3748	1.4572	40
21	1.3121	1.5445	1.3321	1.5136	1.3531	1.4844	1.3752	1.4568	39
22	1.3125	1.5440	1.3324	1.5131	1.3534	1.4839	1.3756	1.4563	38
23	1.3128	1.5434	1.3328	1.5126	1.3538	1.4835	1.3759	1.4559	37
24	1.3131	1.5429	1.3331	1.5121	1.3542	1.4830	1.3763	1.4554	36
25	1.3134	1.5424	1.3335	1.5116	1.3545	1.4825	1.3767	1.4550	35
26	1.3138	1.5419	1.3338	1.5111	1.3549	1.4821	1.3771	1.4545	34
27	1.3141	1.5413	1.3342	1.5106	1.3552	1.4816	1.3774	1.4541	33
28	1.3144	1.5408	1.3345	1.5101	1.3556	1.4811	1.3778	1.4536	32
29	1.3148	1.5403	1.3348	1.5096	1.3560	1.4806	1.3782	1.4532	31
30	1.3151	1.5398	1.3352	1.5092	1.3563	1.4802	1.3786	1.4527	30
31	1.3154	1.5392	1.3355	1.5087	1.3567	1.4797	1.3790	1.4523	29
32	1.3157	1.5387	1.3359	1.5082	1.3571	1.4792	1.3794	1.4518	28
33	1.3161	1.5382	1.3362	1.5077	1.3574	1.4788	1.3797	1.4514	27
34	1.3164	1.5377	1.3366	1.5072	1.3578	1.4783	1.3801	1.4510	26
35	1.3167	1.5371	1.3369	1.5067	1.3581	1.4778	1.3805	1.4505	25
36	1.3170	1.5366	1.3372	1.5062	1.3585	1.4774	1.3809	1.4501	24
37	1.3174	1.5361	1.3376	1.5057	1.3589	1.4769	1.3813	1.4496	23
38	1.3177	1.5356	1.3379	1.5052	1.3592	1.4764	1.3816	1.4492	22
39	1.3189	1.5351	1.3383	1.5047	1.3596	1.4760	1.3820	1.4487	21
40	1.3184	1.5345	1.3386	1.5842	1.3600	1.4755	1.3824	1.4483	20
41	1.3187	1.5340	1.3390	1.5037	1.3603	1.4750	1.3828	1.4479	19
42	1.3190	1.5335	1.3393	1.5032	1.3607	1.4746	1.3832	1.4474	18
43	1.3193	1.5330	1.3397	1.5027	1.3611	1.4741	1.3836	1.4470	17
44	1.3197	1.5325	1.3400	1.5022	1.3614	1.4736	1.3839	1.4465	16
45	1.3200	1.5310	1.3404	1.5018	1.3618	1.4732	1.3843	1.4461	15
46	1.3203	1.5314	1.3407	1.5013	1.3622	1.4727	1.3847	1.4457	14
47	1.3207	1.5309	1.3411	1.5008	1.3625	1.4723	1.3851	1.4452	13
48	1.3210	1.5304	1.3414	1.5003	1.3629	1.4718	1.3855	1.4448	12
49	1.3213	1.5299	1.3418	1.4998	1.3633	1.4713	1.3859	1.4443	11
50	1.3217	1.5294	1.3421	1.4993	1.3636	1.4709	1.3863	1.4439	10
51	1.3220	1.5289	1.3425	1.4988	1.3640	1.4704	1.3867	1.4435	9
52	1.3223	1.5283	1.3428	1.4983	1.3644	1.4699	1.3870	1.4430	8
53	1.3227	1.5278	1.3432	1.4979	1.3647	1.4695	1.3874	1.4426	7
54	1.3230	1.5273	1.3435	1.4974	1.3651	1.4690	1.3878	1.4422	6
55	1.3233	1.5268	1.3439	1.4969	1.3655	1.4686	1.3882	1.4417	5
56	1.3237	1.5263	1.3442	1.4964	1.3658	1.4681	1.3886	1.4413	4
57	1.3240	1.5258	1.3446	1.4959	1.3662	1.4676	1.3890	1.4408	3
58	1.3243	1.5253	1.3449	1.4954	1.3666	1.4672	1.3894	1.4404	2
59	1.3247	1.5248	1.3453	1.4949	1.3669	1.4667	1.3898	1.4400	1
60	1.3250	1.5242	1.3456	1.4945	1.3673	1.4663	1.3902	1.4395	0
′	49° Cosec.	49° Sec.	48° Cosec.	48° Sec.	47° Cosec.	47° Sec.	46° Cosec.	46° Sec.	′

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′	44° Sec.	44° Cosec.	′	′	44° Sec.	44° Cosec.	′	′	44° Sec.	44° Cosec.	′
0	1.3902	1.4395	60	21	1.3984	1.4305	39	41	1.4065	1.4221	19
1	1.3905	1.4391	59	22	1.3988	1.4301	38	42	1.4069	1.4217	18
2	1.3909	1.4387	58	23	1.3992	1.4297	37	43	1.4073	1.4212	17
3	1.3913	1.4382	57	24	1.3996	1.4292	36	44	1.4077	1.4208	16
4	1.3917	1.4378	56	25	1.4000	1.4288	35	45	1.4081	1.4204	15
5	1.3921	1.4374	55	26	1.4004	1.4284	34	46	1.4085	1.4200	14
6	1.3925	1.4370	54	27	1.4008	1.4280	33	47	1.4089	1.4196	13
7	1.3929	1.4365	53	28	1.4012	1.4276	32	48	1.4093	1.4192	12
8	1.3933	1.4361	52	29	1.4016	1.4271	31	49	1.4097	1.4188	11
9	1.3937	1.4357	51	30	1.4020	1.4267	30	50	1.4101	1.4183	10
10	1.3941	1.4352	50	31	1.4024	1.4263	29	51	1.4105	1.4179	9
11	1.3945	1.4348	49	32	1.4028	1.4259	28	52	1.4109	1.4175	8
12	1.3949	1.4344	48	33	1.4032	1.4254	27	53	1.4113	1.4171	7
13	1.3953	1.4339	47	34	1.4036	1.4250	26	54	1.4117	1.4167	6
14	1.3957	1.4335	46	35	1.4040	1.4246	25	55	1.4122	1.4163	5
15	1.3960	1.4331	45	36	1.4044	1.4242	24	56	1.4126	1.4159	4
16	1.3964	1.4327	44	37	1.4048	1.4238	23	57	1.4130	1.4154	3
17	1.3968	1.4322	43	38	1.4052	1.4233	22	58	1.4134	1.4150	2
18	1.3972	1.4318	42	39	1.4056	1.4229	21	59	1.4138	1.4146	1
19	1.3976	1.4314	41	40	1.4060	1.4225	20	60	1.4142	1.4142	0
20	1.3980	1.4310	40								
′	45° Cosec.	45° Sec.	′	′	45° Cosec.	45° Sec.	′	′	45° Cosec.	45° Sec.	′

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4. Frequently Used Angles and Their Functions

Angle	sin A	cos A	tan A	cot A	sec A	csc A
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	1	0	∞	0	∞	1
120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{-\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$
180°	0	-1	0	∞	-1	∞
270°	-1	0	∞	0	∞	-1
360°	0	1	0	∞	1	∞

5. All Functions of an Angle Expressed in Terms of Any One Function

Function	sine	cosine	tangent	cotangent	secant	cosecant
Sin A	$\sin A$	$\pm\sqrt{- \cos^2 A}$	$\frac{\tan A}{\pm\sqrt{1 + \tan^2 A}}$	$\frac{1}{\pm\sqrt{1 + \cot^2 A}}$	$\frac{\pm\sqrt{\sec^2 A - 1}}{\sec A}$	$\frac{1}{\csc A}$
Cos A	$\pm\sqrt{1 - \sin^2 A}$	$\cos A$	$\frac{1}{\pm\sqrt{1 + \tan^2 A}}$	$\frac{\cot A}{\pm\sqrt{1 + \cot^2 A}}$	$\frac{1}{\sec A}$	$\frac{\pm\sqrt{\csc^2 A - 1}}{\csc A}$
Tan A	$\frac{\sin A}{\pm\sqrt{1 - \sin^2 A}}$	$\frac{\pm\sqrt{1 - \cos^2 A}}{\cos A}$	$\tan A$	$\frac{1}{\cot A}$	$\pm\sqrt{\sec^2 A - 1}$	$\frac{1}{\pm\sqrt{\csc^2 A - 1}}$
Cot A	$\frac{\pm\sqrt{1 - \sin^2 A}}{\sin A}$	$\frac{\cos A}{\pm\sqrt{1 - \cos^2 A}}$	$\frac{1}{\tan A}$	$\cot A$	$\frac{1}{\pm\sqrt{\sec^2 A - 1}}$	$\pm\sqrt{\csc^2 A - 1}$
Sec A	$\frac{1}{\pm\sqrt{1 - \sin^2 A}}$	$\frac{1}{\cos A}$	$\pm\sqrt{1 + \tan^2 A}$	$\frac{\pm\sqrt{1 + \cot^2 A}}{\cot A}$	$\sec A$	$\frac{\csc A}{\pm\sqrt{\csc^2 A - 1}}$
Csc A	$\frac{1}{\sin A}$	$\frac{1}{\pm\sqrt{1 - \cos^2 A}}$	$\frac{\pm\sqrt{1 + \tan^2 A}}{\tan A}$	$\pm\sqrt{+ \cot^2 A}$	$\frac{\sec A}{\pm\sqrt{\sec^2 A - 1}}$	$\csc A$

ANSWERS TO PROBLEMS

Paragraph 12.

a(1) $\frac{3}{5}$; .6; 60%. (2) $\frac{1}{2}$; .5; 50%. (3) $\frac{3}{8}$; .375; $37\frac{1}{2}$%. (4) $\frac{1}{4}$; .25; 25%. (5) $\frac{5}{8}$; .625; $62\frac{1}{2}$%. (6) $\frac{3}{5}$; .6; 60%. (7) $\frac{3}{10}$; .3; 30%. (8) $\frac{7}{10}$; .7; 70%. (9) $2\frac{1}{4}$; 2.25; 225%. (10) $\frac{7}{8}$; .875; $87\frac{1}{2}$%. (11) $\frac{2}{25}$; .08; 8%. (12) $\frac{3}{50}$; .06; 6%. (13) $\frac{9}{50}$; .18; 18%. (14) $\frac{1}{400}$; .0025; 25%. (15) $\frac{1}{40}$; .025; $2\frac{1}{2}$%. (16) $\frac{1}{20}$; 05; 5%. (17) $\frac{1}{12}$; $.08\frac{1}{3}$ (See note below); $8\frac{1}{3}$%. (18) $\frac{3}{8}$; .375; $37\frac{1}{2}$%. (19) $1\frac{1}{20}$; 1.05; 105%. (20) $\frac{1}{25}$; .04; 4%.

Note. This mixed decimal and fractional form is often used when an unending decimal would result.

b(1) 150; (2) 50; (3) 4; (4) 900.

c(1) 150%; (2) 275%; (3) 150%; (4) 550%.

d(1) 1.64; (2) 2,496; (3) .34; (4) 4.42.

e(1) .207%; (2) .028%.

f(1) $433\frac{1}{3}$; (2) 2,500; (3) 520; (4) 200; (5) 200.

Paragraph 21.

a 336.6 pounds. b $3\frac{3}{7}$ days. c $5.00. d $1400.00. e .372 ohm. .298 ohm; .459 ohm; .893 ohm. f 2.820 pounds; 3.776 pounds; 4.119 pounds; 2,567 pounds. g 300 rpm. h 157.5 rpm.

Paragraph 25.

a(1) 21; (2) 33; (3) 50; (4) 2.90; (5) 50.1; (6) 70.01; (7) 86.5; (8) 75.89.

b(1) 2.236; (2) 2.646; (3) 3.317; (4) 3.606; (5) 3.873; (6) 4.123.

c(1) .158 ampere; (2) .085 ampere; (3) .283 ampere; (4) 1.118 amperes.

Paragraph 42.

a(1) 17; (2) 58; (3) —21; (4) —139°; (5) —252 volts.

b(1) 251 amperes; (2) —3 volts; (3) —.6875cy; (4) —31.99ax^3; (5) 1.810x^2y.

c(1) —17.92; (2) —72; (3) $\frac{3}{85}$; (4) .075852; (5) .0028125; (6) 120.

d(1) 9; (2) $-\frac{20}{21}$; (3) 700; (4) 250; (5) $-\frac{2}{3}$ ampere; (6) —.0025.

e(1) —4; (2) 14; (3) —25; (4) 19; (5) 11; (6) 16; (7) 44; (8) 66; (9) —46; (10) 18.

Paragraph 50.

a(1) $4a^4 - 4a^2b^2 + 4b^4$. (2) $E + 3RI + 20ZI$. (3) $w + x + 9y + 8z$.

b(1) $19ax + 17by - 9cz$. (2) $-25w - 3x + 8y + 2z$. (3) $4a^2 - 34ab + 6b^2$.

c(1) 7. (2) 1. (3) 1.

d(1) f^{10}. (2) y^{a+b}. (3) v^{2m}. (4) r^5. (5) R^{3m}. (6) r^{m+1}.

e(1) $\frac{4}{x^4}$. (2) $\frac{1}{r^5s^4}$. (3) $\frac{1}{36^ba^{2b}}$. (4) $\frac{1}{I^2R}$. (5) $\frac{a^2}{8b^3}$. (6) $\frac{3E}{4I^2R}$.

f(1) $10a^3b - 15a^2b^2 + 35ab^3$. (2) $4a^3 + 12a^2 + 4a$. (3) $i^3 - 27$. (4) $2x^4 + 5x^3y + 4x^2y^2 + 2xy^3 - y^4$. (5) $9x^4 - 4x^2y^2 +$

$20xy^3 - 25y^4$. (6) $\frac{a-e}{ca}$. (7) $\frac{3L-R^2}{R}$

(8) $1 - 2a^2b + 3a^4b^3$. (9) $2z^2 + z - 1 + \frac{3z+4}{z^2-z+3}$. (10) $4b^2 - b$.

Paragraph 61.

a(1) $5(5 + 1 - 6)$; (2) $4(2 + 1 - 8)$; (3) $3(3 - 6 + 7)$; (4) $7r(1 - 3 + 5)$; (5) $2(5x + 4y + 3z)$.

b(1) $49x^2y^3$; (2) $4w^{10}$; (3) $64a^4b^8$; (4) $729a^9x^3$; (5) $-27b^3z^{12}$.

c(1) 5; (2) -8; (3) $\pm ab^3$; (4) $\pm 6yz^2$; (5) $-10ab^5$; (6) $\pm 20a$; (7) -3; (8) $-x^3$; (9) 4; (10) $5x^4y^5z^2$.

d (1) $3(x + 2)$; (2) $5a(a + 3)$; (3) $2x(5x^2 - 7x - 1)$; (4) $3z(2ay + 3bx - 4c)$; (5) $m(m^2 + m - 5x)$; (6) $3a^3(a^2 - 2ab - b^2)$; (7) $7ry^3(1 - 2 + 3)$ or 14^3ry; (8) $2xam(6x + 7a + 8m)$; (9) $\pi(r_1^2 + r_2^2)$; (10) $\frac{1}{16}cd(4c^2 - 2cd + d^2)$.

Paragraph 69.

a(1) $x = 5\frac{2}{5}$; (2) $x = 4$; (3) $r = 2$; (4) $x = -1\frac{1}{12}$; (5) $t = 1$; (6) $x = 7\frac{3}{4}$; (7) $r = 4$; (8) $x = 1$.

b(1) 8; (2) x; (3) $3(r + s)$; (4) $3(a - s)$; (5) $(I - 6)(I - 9)$; (6) $\frac{8E^2I^2}{2I^2R}$; (7) $\frac{2f}{6\pi f^2c}$.

c(1) $\frac{rR}{rR^2}, \frac{r}{rR^2}, \frac{R^2}{rR^2}$; (2) $\frac{a-1}{a^2-1}, \frac{x(a+1)}{a^2-1}$; (3) $\frac{3b}{6x}, \frac{2c}{6x}$; (4) $\frac{y(y+3)}{2(y+3)}, \frac{y}{2(y+3)}$; (5) $\frac{2(c+1)}{c(c+1)}, \frac{3c}{c(c+1)}$; (6) $\frac{2i}{2e-10}, \frac{i}{2e-10}$; (7) $\frac{y}{C^2-d^2}, \frac{z(c+d)}{C^2-d^2}$.

d(1) $\frac{12}{a}$; (2) $\frac{7s+11}{4t}$; (3) $\frac{9y^2a+10xb}{12x^2y^3}$; (4) $\frac{6(z^2-2)}{z^4-5z^2+14}$; (5) $\frac{9c+2cd-12d}{12c^2d^2}$; (6) $\frac{2r^2+r-13}{r^2+2r-15}$; (7) $\frac{12y-1}{4}$; (8) $\frac{4ab}{a^2-b^2}$; (9) $\frac{16(2-5q)}{25q^2}$; (10) $\frac{3t+4y}{12tv^2}$

e(1) $\frac{3y^2}{8}$; (2) $\frac{a^9}{b^6}$; (3) $\frac{xz}{21my}$; (4) $\frac{(s-r)^2}{s^2}$; (5) $\frac{3}{5x}$; (6) $\frac{1}{a^3}$; (7) $15z$; (8) $\frac{a^3}{6cd}$; (9) $\frac{4su}{5}$; (10) $\frac{e+3}{e+2}$.

Paragraph 76.

a(1) 2; (2) 16; (3) $5\sqrt{2}$; (4) $\frac{\sqrt[3]{4}}{4}$; (5) $3\sqrt{2x-1}$; (6) $\frac{x^3\sqrt[n]{6}}{y}$; (7) x^2y; (8) $d^{\frac{9}{2}}e^3$; (9) $\frac{4r^2}{s}$; (10) a^3b.

b(1) $\sqrt[3]{4}$; (2) $\sqrt[6]{a^9b^4}$; (3) $\sqrt[3]{6^2}$; (4) $\sqrt[2]{2f}$; (5) $\sqrt[5]{x}$; (6) $\sqrt[4]{a^3b^6}$; (7) $6\sqrt[3]{r}$; (8) $26\sqrt[3]{a^2}$; (9) $\sqrt{2r_1 + 3r_2}$; (10) $8y\sqrt[8]{x}$.

c(1) $a^{\frac{1}{2}}$; (2) $(5x)^{\frac{1}{3}}$; (3) $6xd^{\frac{1}{3}}$; (4) $z^{\frac{1}{2}}$; (5) $(3a^3b^5)^{\frac{1}{2}}$; (6) $y^3a^{\frac{1}{2}}$; (7) $8(3e)^{\frac{1}{3}}$; (8) $9g^{\frac{1}{3}}$; (9) $3bc^{\frac{1}{2}}d^{\frac{1}{3}}$; (10) $(x - y)^{\frac{1}{2}}$.

d(1) $2\sqrt{3}$; (2) $3\sqrt{7}$; (3) $3x\sqrt{7}$; (4) $12ab^2\sqrt{2}$; (5) $2bd\sqrt{15}$; (6) $2I\sqrt{2R}$; (7) $9pz\sqrt{7p}$; (8) $12dr^4s\sqrt{3ds}$; (9) $45a^2\sqrt{b}$; (10) $112w^4x^2y\sqrt{2xz}$.

e(1) $\frac{\sqrt{2}}{10}$; (2) $\frac{\sqrt{x}}{2x}$; (3) $\frac{\sqrt[3]{3a}}{3}$; (4) $\frac{\sqrt[3]{x^2}}{x}$; (5) $\frac{\sqrt[4]{27a^3x^2}}{3ax}$; (6) $\frac{\sqrt[3]{(3-2x)^2}}{3-2x}$; (7) $\frac{\sqrt[3]{a}\,(a+b)}{a}$; (8) $\frac{\sqrt[3]{ab^2c^2}}{bc}$; (9) $\frac{\sqrt[3]{s+1}}{s+1}$; (10) $\sqrt[5]{(i+3)^3}$.

f(1) 10; (2) $14\sqrt{5}$; (3) $x - \frac{x\sqrt{3}}{2}$; (4) $\frac{3a\sqrt{2} + a}{2}$; (5) $(r + 1)\sqrt{rst}$; (6) $\frac{2y\sqrt{x^2 - y^2}}{x^2 - y^2}$; (7) $\sqrt[3]{5} + 8\sqrt{x}$; (8) $7\sqrt{a} - 6\sqrt{b}$; (9) $3\sqrt{x + y} - 4\sqrt{x - y}$; (10) $7ab\sqrt{5a}$.

g(1) $12\sqrt{10}$; (2) 18; (3) $8ab^2$; (4) $2z^3\sqrt{3z}$; (5) $2xy\sqrt[5]{xy}$; (6) $24pq^2r\sqrt[3]{qr^2}$; (7) $a + b + c + 2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$; (8) $ax\sqrt{a}\ (a + x + 1)$; (9) 8; (10) $2axy^2\sqrt[3]{2a}$.

h(1) 2; (2) 5; (3) $2\sqrt[3]{x}$; (4) $3\sqrt{zy}$; (5) $\sqrt{6} + 2$; (6) $12a\sqrt[12]{2^3 \cdot 3^5 \cdot 5^4 \cdot a^2}$; (7) $\frac{c - \sqrt{2c} - 4}{c - 8}$ (8) $\sqrt{15}$; (9) $\frac{e^2 + f^2 + 2f\sqrt{e^2 \quad f^2}}{e^2}$; (10) $\frac{4b\sqrt{1 - 4b^2} + 1}{8b^2 - 1}$

Paragraph 79.

a(1) $j5\sqrt{3}$; (2) $j\sqrt{\quad}$; (3) $-j8x^2\sqrt{ax}$; (4) $-j10x^2y^2\sqrt{x}$; (5) $\frac{\quad}{3}$; (6) $-4xy\sqrt[3]{2x^2y^2}$.

b(1) 16 + j109; (2) 41 — j22; (3) 61— j251; (4) 4 + j10: (5) 6 + j11; (6) —2 — j47.

c(1) 779 — j371; (2) 59 + j114; (3) —22 + j15; (4) 155 — j61; (5) 169 + j23; (6) 9 — j8.

d(1) —55 + j46; (2) $6 - 6\sqrt{6} + j(6\sqrt{2} + 6\sqrt{3})$; (3) 13; (4) —5 — j12; (5) — j8; (6) 46 — j48; (7) $f^2 + jfg - g^2$; (8) $I^2 + E^2$; (9) —68 — j239; (10) 71 — j17.

e(1) $\frac{3}{13} - j\frac{2}{13}$; (2) 1 — j6; (3) $-\frac{6}{25} + j\frac{17}{25}$; (4) 1 + j2; (5) $\frac{x^2 + j^2xy - y^2}{x^2y^2}$; (6) 2(1 — j2); (7) $\frac{3(1 + j)}{2}$; (8) $\frac{1 + j13}{10}$; (9) $\frac{38 + j34}{65}$; (10) $\frac{I^2 + j2IE - E^2}{I^2 \quad E^2}$

Paragraph 86.

a(1) 3; (2) 2; (3) 85; (4) 3; (5) 1; (6) $x = -5$, $y = 8$; (7) $a = 3$, $b = 1$; (8) $x = 3$, $y = 4$; (9) $m = 3$, $n = 5$; (10) $r = 8$, $s = 1$

b(1) $d = \frac{Wh}{F}$ (2) $g = \frac{v^2 - v_o^2}{2h}$. (3) $a = \frac{Fg}{w}$. (4) $N = \frac{2.534H}{D^2}$. (5) $l = \frac{10^8F}{22.5BI}$.

c(1) 15; (2) 0; (3) $\frac{10}{3}$; (4) 4; (5) $\frac{28}{9}$; (6) $\frac{12}{119}$; (7) $-2\frac{12}{25}$; (8) 8; (9) $\frac{40}{109}$; (10) $-\frac{1}{19}$.

d(1) $x = 4$, $y = 5$; (2) $a = 4.95$, $b = 2.61$; (3) $x = 4$, $y = 7$; (4) $x = -2$, $y = -4$; (5) $x = -3$, $y = 1$; (6) $I = 13$, $Z = 17$; (7) $x = 4$, $y = \frac{1}{2}$; (8) $a = 6$, $b = -4$; (9) $x = 5$, $y = -1$; (10) $r = \frac{(a + b)}{2}$, $s = \frac{(a - b)}{2}$

e(1) 1 volt; (2) R — 20 ohms; (3) 110 volts; (4) 75 ohms; (5) 100 milliamperes, 80 milliamperes, 60 milliamperes; (6) 5.5 amperes.

Paragraph 94.

a(1) 0, $-\frac{3}{2}$; (2) 0, 4; (3) 0, —3; (4) 0, —2; (5) ±8; (6) ±3; (7) ±3; (8) ±4; (9) 7, —6; (10) 1, 12.

b(1) $\frac{-3 \pm \sqrt{13}}{2}$;(2) $-3 \pm \sqrt{19}$; (3) $2 \pm \sqrt{3}$; (4) $-2 \pm \frac{\sqrt{22}}{2}$; (5) $\frac{1}{2} \pm \frac{\sqrt{14}}{4}$; (6) $-\frac{5}{3} \pm \frac{2\sqrt{10}}{3}$; (7) —1, 3; (8) $-1 \pm \frac{\sqrt{6}}{2}$; (9) $1 \pm \sqrt{6}$; (10) $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$e(1) -1; (2) -\frac{3}{4}, \frac{2}{3}$; (3) $\frac{-5 \pm \sqrt{13}}{6}$; (4) $\frac{3}{2}, \frac{4}{3}$; (5) $-3, 1$; (6) $-\frac{1}{5}, \frac{5}{3}$; (7) $\pm \sqrt{2}$; (8) $\pm \sqrt{19}$; (9) $-1, 2$; (10) $\frac{-5 \pm \sqrt{7}}{3}$

Paragraph 111.

a(1) $1{,}613 \times 10^{3}$; (2) 500×10^{3}, or 5×10^{5}; (3) $6{,}166 \times 10^{-6}$.

b(1) $3{,}109 \times 10^{2}$; (2) 19×10^{-4}; (3) $4{,}492 \times 10^{-4}$.

c(1) 892×10^{3}; (2) $2{,}464 \times 10^{-2}$, or 24.64; (3) $3{,}168 \times 10^{-11}$; (4) 14,640.

d(1) 167; (2) $1{,}608 \times 10^{7}$; (3) 107; (4) 33×10^{-5}.

e(1) 4×10^{2}, or 400; (2) 13×10^{-4}; (3) 27×10^{-9}; (4) 9×10^{2}, or 900.

Paragraph 127.

a(1) 2.8949; (2) 0.5527; (3) 8.5378–10; (4) 6.6776–10; (5) 1.6955; (6) 2.4370; (7) 2.8809; (8) 0.8593; (9) 7.9946–10; (10) 5.7205–10.

b(1) 70,100; (2) 271; (3) .351; (4) .000676; (5) 3.99; (6) 370.67; (7) .00002718; (8) 500,500; (9) 1.5915; (10) .000003445.

c(1) 164.2; (2) 39,982; (3) 1,376; (4) .006764; (5) 5,710.

d(1) .4983; (2) .3874; (3) .3984; (4) .7487; (5) .2437.

e(1) .0000007372; (2) 51.46; (3) 3.47; (4) 19.43; (5) 783; (6) .2367; (7) 5.343; (8) 87.74; (9) 1.55; (10) .09456.

f(1) 2.298; (2) 11.77; (3) 24.43; (4) 83.28; (5) .4509; (6) .4725; (7) .04088; (8) .6153; (9) .0576; (10) .35367.

Paragraph 142.

a 96 square inches. *b* 36 square inches. *c* 25 square inches. *d* 15 square inches. *e* 14.422 square inches. *f* 5.657 square inches. *g*(1) Parallelogram, $A = bh$, 120 square inches; (2) Triangle, $A = \frac{bh}{2}$ 4.025 square inches; (3) Circle, $A = \pi r^2$, 314 square centimeters, $C = \pi D$, 62.8 centimeters; (4) Trapezoid, $A = \frac{B + b}{2} h$, $A = 60$ square inches. *h*(1) 3 inches; (2) $4\frac{1}{2}$ inches; (3) 8.8 inches; (4) 5 inches. *i* 78.5 square inches. *j* 100 feet. *k* 82.5 square feet. *l* 48.496 inches.

Paragraph 153.

a(1) $c = 8.603$. (2) $a = 6.08$. (3) $b = 39.5$. (4) $c = b\sqrt{10}$. (5) $b = m^2 - 1$.

b(1) $\sin A = \frac{4}{7}$, $\cos A = \frac{\sqrt{33}}{7}$, $\tan A = \frac{4}{\sqrt{33}}$, $\cot A = \frac{\sqrt{33}}{4}$, $\sec A = \frac{7}{33}\sqrt{33}$, $\csc A = \frac{7}{4}$

(2) $\sin A = \frac{2}{13}\sqrt{13}$, $\cos A = \frac{3}{13}\sqrt{13}$, $\tan A = \frac{2}{3}$, $\cot A = \frac{3}{2}$, $\sec A = \frac{\sqrt{13}}{3}$, $\csc A = \frac{\sqrt{13}}{2}$

(3) $\sin A = \frac{1}{2}$, $\cos A = \frac{\sqrt{3}}{2}$, $\tan A = \frac{\sqrt{3}}{3}$, $\cot A = \sqrt{3}$, $\sec A = \frac{2}{3}\sqrt{3}$, $\csc A = 2$.

(4) $\sin A = \frac{1}{2.4}$, $\cos A = \frac{1.09}{1.2}$, $\tan A = \frac{1}{2.18}$, $\cot A = 2.18$, $\sec A = \frac{1.2}{1.09}$, $\csc A = 2.4$.

(5) $\sin A = y\frac{\sqrt{y^2+1}}{y^2+1}$, $\cos A = \frac{\sqrt{y^2+1}}{y^2+1}$, $\tan A = y$, $\cot A = \frac{1}{y}$, $\sec A = \sqrt{y^2+1}$, $\csc = \frac{\sqrt{y^2+1}}{y}$.

(6) $\sin A = \frac{\sqrt{55}}{8}$, $\cos A = \frac{3}{8}$, $\tan A = \frac{\sqrt{55}}{3}$, $\cot A = \frac{3\sqrt{55}}{55}$, $\sec A = 2\frac{2}{3}$, $\csc A = \frac{8\sqrt{55}}{55}$

c(1) $a = 17$, $b = 29.4$, $c = 34$. (2) $a = 9$, $b = 12$, $c = 15$. (3) $a = 12$, $b = 16$, $c = 20$. (4) $a = 17.5$, $b = 10\sqrt{11}$, $c = 37.5$. (5) $a = 10$, $b = 6$, $c = 2\sqrt{34}$. (6) $a = 37.08$, $b = 18.4$, $c = 41.4$.

d(1) $b = 10\sqrt{3}$, $c = 20$. (2) $a = 7$, $c = 7\sqrt{2}$. (3) $a = 4\sqrt{3}$, $b = 4$. (4) $b = 3\sqrt{3}$, $c = 6\sqrt{3}$. (5) $a = 12.5$, $b = 12.5\sqrt{3}$.

Paragraph 164.

a(1) .02618, .99966, .02619, 38.1885. (2) .26584, .96402, .27576, 3.62636. (3) .53238, .84650, .62892, 1.59002. (4) .59693, .80230, .74402, 1.34405. (5) .70690, .70731, .99942, 1.00058. (6) .70706, .70716, .99986, 1.00014. (7) .57649, .81710, .70553, 1.41737. (8) .81370, .58129, 1.39982, .71438. (9) .74811, .66357, 1.12740, .88700. (10) .92429, .38169, 2.42158, .41295.

b(1) 14° 54′ 51″; (2) 66° 35′ 51″; (3) 19° 56′ 54″; (4) 25° 17′ 5″; (5) 40° 23′ 35″; (6) 68° 45′ 2″; (7) 22° 11′ 47″; (8) 34° 5′ 19″; (9) 52° 13′ 2″; (10) 51° 29′ 49″

c(1) 44° 43′ 29″; (2) 10.29; (3) 32.9; (4) 19.76; (5) 12.4; (6) 54° 18′ 52.5″; (7) 33.69; (8) 16.5; (9) 36° 28′ 9″; (10) 128.3; (11) 32.9; (12) 29° 3′ 15″

d(1) 43.845 feet; (2) 80.027 feet; (3) 12.226 feet, 8.69 feet high; (4) 3,149 feet; (5) 11.734 feet; (6) 91.77 feet; (7) 206 feet; (8) 3,578 feet; (9) 16,647 feet (3.153 miles); (10) 82.12 feet; (11) 1.414 inches each; (12) side opposite 60° ∠ 5.196 inches, side opposite 30° ∠ 3 inches.

Paragraph 173.

a $C = 62° 16′ 38″$, $a = 14.59$. *b* $B = 69° 58′$. *c* $A = 23° 33′ 22″$, $B = 45° 16′ 31″$, $C = 111° 10′ 7″$. *d* $A = 81° 31′ 41″$, $B = 41° 7′ 29″$. *e* 240 square inches. *f* 97.880 square feet. *g* 55.424 square inches. *h* $A = 32° 33′ 45″$, $B = 84° 36′ 15″$, $c = 15.95$ inches.

Paragraph 176.

a(1) .4 radian; (2) 4 radians; (3) 8 radians; (4) 2.78 radians.

b(1) 35 inches; (2) 17.6 feet; (3) 18.9 miles; (4) .00198 inch.

c(1) .52 radian; (2) 4.6 radians; (3) 2.77 radians; (4) 5.89 radians.

d(1) 45° 50′ 11.8″; (2) 1432° 23′ 40.2″ (3) 197° 40′ 13.44″; (4) 540°.

e(1) $\pi/6$; (2) $\pi/3$; (3) $5\pi/4$; (4) 4π.

Paragraph 192.

a(1) 5.5 amperes; (2) 1.30 amperes.

b(1) 28 to 100 ohms; (2) 7 to 25 ohms.

c 221 volts; 7.514 watts.

d(1) .8 ampere; (2) 24 volts; (3) 80 volts.

e(1) G_T = .35 inch; (2) 2.857 ohms; (3) $I_2 = 10$ amperes, $I_3 = 5$ amperes; (4) $I_T = 35$ amperes.

f(1) $G_1 = 1$ mho, $G_2 = .333$ mho, $G_3 = .1$ mho, $G_4 = .05$ mho, $G_5 = .02$ mho; (2) $G_T = 1.503$ mhos; (3) $R_T = .665$ mho.

g(1) 17.08 ohms; (2) 86.4 volts; (3) 100.8 volts; (4) 228.192 volts.

h(1) 4.62 ohms; (2) 5.359 ohms; (3) 4.783 ohms; (4) 15.246 ohms.

i(1) 125 volts; (2) $E_1 = 50$ volts, $E_2 = 75$ volts.

j(1) 10.754 volts; (2) $I_1 = .7028$ ampere, $I_2 = 8.269$ amperes, $I_3 = 1.0237$ amperes;

(3) $I_1R_1 = 10.753$ volts, $I_2R_2 = 10.750$ volts, $I_3R_3 = 10.749$ volts.

k(1) 5,500 amperes; (2) .22729 ampere.

l A, 5 ohms; B, $6\frac{2}{5}$ ohms; C, 8 ohms.

m A, 3 ohms; B, 12 ohms; C, 7 ohms; D, 25 ohms.

n A, $8\frac{8}{9}$ ohms; B, $8\frac{1}{3}$ ohms; C, 10 ohms.

o A, 3 ohms; B, 4 ohms; C, 2 ohms.

p $I_{R1} = 7.519$ amperes; $I_{R2} = 3.214$ amperes; $I_{R3} = 1.176$ amperes; $I_{R4} = I_{R7} = I_{R8} = 0.392$ ampere; $I_{R5} = I_{R10} = 4.782$ amperes; $I_{R6} = I_{R9} = 1.568$ amperes, $I_{R11} = 12.301$ amperes.

q $R = 239.1688$ ohms.

r $I = .457$ ampere.

Paragraph 204.

a(1) 94 ohms; (2) 184 ohms; (3) current leads voltage because capacitive reactance exceeds inductive reactance; (4) 104 ohms; (5) 5 amperes.

b. 9.425 ohms.

c. 63,662 ohms (approx).

d. 80 ohms.

e. 455 kc.

f(1) 30 amperes; (2) 180 volts; (3) 480 volts; (4) 240 volts.

g(1) 3 amperes; (2) 4 amperes; (3) 6 amperes; (4) 83 ohms; (5) 34°; (6) 896.4 watts; (7) 1080 watts, or 1.08 kw

Paragraph 208.

a. 39.8 to 1.

b. —.744 db per mile.

c. +10.8 db.

d. 25.1 to 1.

Paragraph 217.

a. 5%.

b. 77.7%

c. 15%.

d. 12,600 ohms.

e. 86.7%.

f. 2,200 watts.

g. 1.58 amperes.

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11-597

NG: State AG (3); units—same as Active Army except allowance is one copy to each unit.

USAR: None.

For explanation of abbreviations used, see AR 320-50.

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